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Closed-form Solutions for an Explicit Modern Ideal Tontine with Bequest Motive.

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Abstract

In this paper I extend the work of Bernhardt and Donnelly (2019) dealing with modern explicit tontines, as a way of providing income under a specified bequest motive, from a defined contribution pension pot. A key feature of the present paper is that it relaxes the assumption of fixed proportions invested in tontine and bequest accounts. In making the bequest proportion an additional control function I obtain, hitherto unavailable, closed-form solutions for the fractional consumption rate, wealth, bequest amount, and bequest proportion under a constant relative risk averse utility. I show that the optimal bequest proportion is the product of the optimum fractional consumption rate and an exponentiated bequest parameter. Typical scenarios are explored using UK Office of National Statistics life tables, showing the behaviour of these characteristics under varying degrees of constant relative risk aversion.

1. Explicit Modern Tontines

In a recent paper, Bernhardt and Donnelly (2019) model an explicit modern and ideal tontine with a bequest motive. It is ideal in that it assumes that there are an infinite number of members of the tontine. It is explicit in the sense that a member receives in the interval $(t, t + dt)$ a longevity credit of $\lambda(t)X(t)dt$ where $\lambda(t)$ is the mortality rate and $X(t)$ is the size of the pension pot at age t. The longevity credit arises through the death of other members of the group in that time interval. When the individual dies his/her pension pot is distributed to other members of the tontine in such a way that the scheme is actuarially fair. A distinctive feature of the Bernhardt and Donnelly paper is that it includes a bequest motive. Thus a proportion $1 - \alpha$ of the current pension pot is assigned to a bequest account that does not attract longevity credits, while the remaining proportion α is assigned to the tontine account. On the death of the member, the tontine account is shared out amongst members of the tontine. In contrast, the bequest account passes to the estate of the member. With such a bequest motive, the longevity credit becomes $\alpha\lambda(t)X(t)dt$. The problem is to determine an optimum fractional consumption rate, $c(t)$, expressed as a proportion of the pension pot value (wealth), $X(t)$. At all times the amounts in the tontine and bequest accounts are respectively $\alpha X(t)$ and $(1 - \alpha)X(t)$. A proportion 1-$w(t)$ of the pension pot is invested in a riskless asset which grows at rate r and the remaining proportion $1 - w(t)$ is invested in a risky asset whose return in $(t, t + dt)$ is $(\mu - 0.5\sigma^2)dt + \sigma dW(t)$ where $\mu > r$ and $\{W(t)\}$ is the Wiener process. Because of the risky asset, transfers between the tontine and bequest accounts must be made continuously to maintain the proportion α.

The individual chooses $\{c(t)\}$ in such a way as to maximize the expected utility over a lifetime. Specifically, the utility gained through consumption (withdrawal) in $(t, t + dt)$ is $U[c(t)X(t))]dt$ while the contribution to the individual's utility through death at age τ say, is $bU[(1 - \alpha)X(\tau)]$ where $b > 0$ is a parameter that expresses the strength of the bequest motive.

Following the approaches of Merton (1970), Richard (1975), and others, the utility function chosen is from the constant relative risk aversion family with a power utility function

$$U(x) = \frac{x^\gamma}{\gamma} \tag{1}$$

for $\gamma < 1 | \neq 0$. This has a constant relative risk aversion of $1 - \gamma$.

2. Derivation of Optimal Consumption

With this background and with a time preference rate of ρ applied to utilities accruing at future times, I define a value function, similar to that in Bernhardt and Donnelly (2019), namely

$$V(t,x) = \max_{\{w(s),\alpha(s),c(s):s\geq t\}} \mathbb{E}\left(\int_t^\infty \exp\left[-\int_t^s [\lambda(u) + \rho]du\right]\{U[c(s)X(s)] + b\lambda(s)U[(1-\alpha)X(s)]\}ds \Big| X(t) = x\right) \tag{2}$$

where the pension pot value (wealth function) follows

$$\frac{dX(t)}{X(t)} = \left(r + (\mu - r)w(t) + \alpha(t)\lambda(t) - c(t)\right)dt + \sigma w(t)dW(t) \tag{3}$$

and where $\{W(t)\}$ is the Wiener process. The control functions are constrained so that $w(s) \geq 0, 1 \geq \alpha(s) \geq 0, c(s) \geq 0$ for all $s \geq t$. The proportion of funds invested in the risky asset might exceed 1. I allow for this possibility by assuming, for theoretical expediency, that borrowing is possible at the risk-free rate r. Replacing the utility (1) by $U(x) = (x^\gamma - 1)/\gamma$ in (2) will yield the same optimal control values. Since $\lim_{\gamma \to 0} \frac{x^\gamma - 1}{\gamma} = \ln x$, we identify this as the utility function for this limiting case.

A significant departure from the approach of Bernhardt and Donnelly (2019) is that I replace their constant α by an age dependent control function $\alpha(t)$. There are two reasons for this. Firstly, choosing α dynamically will lead to a higher expected utility. Secondly, we shall see that under the reasonable assumption that

$$b^{\frac{1}{1-\gamma}}\left[r + \frac{\rho-r}{1-\gamma} - \frac{\gamma}{2}\left(\frac{\mu-r}{\sigma}\right)^2\left(\frac{1}{1-\gamma}\right)^2\right] < 1 \tag{4}$$

this leads to a closed-form solution for $c(t)$ and $X(t)$, whereas the constant α case does not.[1]

Following Bernhardt and Donnelly (2019), the optimum strategy is obtained using a dynamic programming Hamilton-Jacobi-Bellman approach, by solving

$$[\lambda(t) + \rho]V(t,x) = \max_{w,c,\alpha}\left\{\frac{(cx)^\gamma}{\gamma} + b\lambda(t)\frac{[(1-\alpha)x]^\gamma}{\gamma} + \frac{\partial V(t,x)}{\partial t} + x\frac{\partial V(t,x)}{\partial x}[r + w(\mu - r) + \alpha\lambda(t) - c] + \frac{1}{2}x^2\sigma^2 w^2\frac{\partial^2 V(t,x)}{\partial x^2}\right\}^2$$

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[1] For the logarithmic utility but not for the general power utility, for the case where α is fixed rather than dynamically controlled, Bernhardt and Donnelly (2019) obtain a closed-form solution for $c(t)$ but not for $X(t)$.

$$(5)$$

where I have now augmented the optimization to include the control α.

Taking the first and second derivatives of the expression in braces with respect to c gives a maximum at

$$(c^*x)^{\gamma-1}x - x\frac{\partial V}{\partial x} = 0 \tag{6}$$

which, with the hope that c^* is independent of x, is suggestive of a trial solution of the form

$$V(t,x) = \frac{c^{*\gamma-1}x^\gamma}{\gamma}. \tag{7}$$

The first order condition on w gives

$$x\frac{\partial V}{\partial x}(\mu - r) + wx^2\sigma^2\frac{\partial^2 V}{\partial x^2} = 0. \tag{8}$$

From (7), $\frac{\partial^2 V}{\partial x^2} < 0$, which implies that the stationary point is a maximum, and that consequently

$$w^*(t)(\mu - r) = \left(\frac{1}{1-\gamma}\right)\left(\frac{\mu-r}{\sigma}\right)^2. \tag{9}$$

Note that $w^*(t)$ is independent of t and so henceforth is expressed as w^*. Further, when $\gamma > 1 - \frac{\mu-r}{\sigma^2}$, then $w^* > 1$. In that case, borrowing to allow over investment in the risky asset raises the probability of running down the pension pot to near zero at an early age, alongside a small probability of large wealth at an advanced age. These aspects are consistent with a low relative risk aversion, 1-γ. Finally, the first order condition on α is

$$-b\lambda(t)x[(1-\alpha)x]^{\gamma-1} + \lambda(t)x\frac{\partial V(t,x)}{\partial x} = 0 \tag{10}$$

And, differentiating again, it is seen that this is a maximum. Using (7) this gives

$$1 - \alpha^*(t) = \min\left[c^*(t)b^{\frac{1}{1-\gamma}}, 1\right]. \tag{11}$$

Let us assume that b is of a magnitude such that $c^*(t)b^{\frac{1}{1-\gamma}} < 1$ for all t. Substituting (7), (9), and (11) into (5)

$$[\lambda(t) + \rho]V(t,x) = c^*(t)V(t,x) + \lambda(t)c^*(t)V(t,x)b^{\frac{1}{1-\gamma}} + V(t,x)(\gamma - 1)\frac{1}{c^*(t)}\frac{dc^*(t)}{dt} +$$
$$\gamma V(t,x)[r + w^*(\mu - r) + \alpha(t)\lambda(t) - c^*(t)] + \frac{V(t,x)}{2}\gamma(\gamma - 1)\sigma^2 w^{*2},$$

$$(12)$$

that is

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[2] Bernhardt and Donnelly (2019) suggest a value function $V(t,x) = h(t)x^\gamma$ and $c(t) = (\gamma h(t))^{\frac{1}{1-\gamma}}$ but there appears to be an error in that these do not satisfy their Chini equation labelled equation (9) in their paper. However, if $c(t) = (\gamma h(t))^{\frac{1}{\gamma-1}}$ that does lead to their equation (9).

$$\frac{1}{c^*(t)}\frac{dc^*}{dt} = c^*(t)\left[1 + b^{\frac{1}{1-\gamma}}\lambda(t)\right] - k(t) \tag{13}$$

where

$$k(t) = r + \lambda(t) + \frac{\rho-r}{1-\gamma} - \frac{\gamma}{2}\left(\frac{\mu-r}{\sigma}\right)^2\left(\frac{1}{1-\gamma}\right)^2. \tag{14}$$

Equation (13) is a Bernoulli differential equation with solution

$$c^*(b,\gamma,t) = \frac{1}{m(b,\gamma,t)} \tag{15}$$

where

$$m(b,\gamma,t) = \int_t^\infty [1 + b^{\frac{1}{1-\gamma}}\lambda(u)]e^{-[K(u)-K(t)]}du \tag{16}$$

and where $K(t) = \int_0^t k(y)dy$. Now define

$$\beta = k(t) - \lambda(t) = r + \frac{\rho-r}{1-\gamma} - \frac{\gamma}{2}\left(\frac{\mu-r}{\sigma}\right)^2\left(\frac{1}{1-\gamma}\right)^2. \tag{17}$$

Given that $\lambda(t)$ is increasing in t,

$$m(b,\gamma,t) = \int_t^\infty [1 + b^{\frac{1}{1-\gamma}}k(u)]e^{-[K(u)-K(t)]}du - b^{\frac{1}{1-\gamma}}\beta \int_t^\infty e^{-[K(u)-K(t)]}\,du$$

$$= b^{\frac{1}{1-\gamma}} + \left[1 - b^{\frac{1}{1-\gamma}}\beta\right]\int_t^\infty e^{-[K(u)-K(t)]}\,du = b^{\frac{1}{1-\gamma}} + \left[1 - b^{\frac{1}{1-\gamma}}\beta\right]m(0,\gamma,t).$$

$$\tag{18}$$

Therefore

$$c^*(b,\gamma,t) = \frac{1}{b^{\frac{1}{1-\gamma}}+\left(1-b^{\frac{1}{1-\gamma}}\beta\right)m(0,\gamma,t)}. \tag{19}$$

Now

$$c^*(b,\gamma,t)b^{\frac{1}{1-\gamma}} = \frac{b^{\frac{1}{1-\gamma}}}{b^{\frac{1}{1-\gamma}}+\left[1-b^{\frac{1}{1-\gamma}}\beta\right]m(0,\gamma,t)} \tag{20}$$

and so assuming that $1 - b^{\frac{1}{1-\gamma}}\beta > 0$, as in equation (4), then $c^*(b,\gamma,t)b^{\frac{1}{1-\gamma}} < 1$, which was the assumption made following result (11). The condition $1 - b^{\frac{1}{1-\gamma}}\beta > 0$ is not restrictive for most real-world situations.

Note that with a logarithmic utility function, we set $\gamma = 0$, giving $\beta = \rho$ and $k(t) = \lambda(t) + \rho$. Thus when $b < \rho^{-1}$ we find that

$$c^*(b,0,t) = \frac{1}{b+[1-b\rho\,]m(0,0,t)} = \frac{1}{b+[1-b\rho\,]\int_t^\infty e^{-\int_t^u[\lambda(y)+\rho]dy}du} \tag{21}$$

which is, perhaps surprisingly, identical to result (14) in Bernhardt and Donnelly (2019), who optimise for a fixed rather than dynamic α, and dynamic $c(b,0,t)$. According to result (11), the dynamic proportion invested in the bequest account is simply $bc^*(b,0,t)$.

3. Evolution of Wealth, Discounted Bequest Value and Discounted Monetary Consumption rate

Returning to the general power utility, I now obtain a closed-form solution for the wealth equation under the combined optimal consumption, bequest proportion, and risk/riskless proportion. Using the fact that when $1 - b^{\frac{1}{1-\gamma}}\beta > 0$ we have from result (11) that $1 - \alpha^*(t) = c^*(t)b^{\frac{1}{1-\gamma}}$, then

$$\frac{dX(t)}{X(t)} = \left(r + (\mu - r)w^* + \left[1 - c^*(t)b^{\frac{1}{1-\gamma}}\right]\lambda(t) - c^*(t)\right)dt + \sigma w^* dW(t)$$

$$= \left(r + (\mu - r)w^* + \lambda(t) - c^*(t)\left[1 + b^{\frac{1}{1-\gamma}}\lambda(t)\right]\right)dt + \sigma w^* dW(t).$$

$$(22)$$

But from (13)

$$c^*(t)\left[1 + b^{\frac{1}{1-\gamma}}\lambda(t)\right] = \frac{d\ln[c^*(t)]}{dt} + k(t) \tag{23}$$

and so

$$\frac{dX(t)}{X(t)} = \left(r + (\mu - r)w^* + \lambda(t) - \frac{d\ln[c^*(t)]}{dt} - k(t)\right)dt + \sigma w^* dW(t)$$

$$= \left(r - \beta + (\mu - r)w^* - \frac{d\ln[c^*(t)]}{dt}\right)dt + \sigma w^* dW(t)$$

$$(24)$$

where $c^*(t) = c^*(b, \gamma, t)$.

Solving this subject to an initial condition that $X(s) = x_s$ we obtain for $t > s$

$$X(t) = \frac{x_s c^*(s)e^{\left(r - \beta + (\mu - r)w^* - 0.5\sigma^2 w^{*2}\right)(t-s)}e^{\sigma w^* W(t-s)}}{c^*(t)} \tag{25}$$

and

$$\mathbb{E}(X(t)) = \frac{x_s c^*(s)e^{(r - \beta + (\mu - r)w^*)(t-s)}}{c^*(t)}. \tag{26}$$

The stochastic bequest amount, discounted by the time preference rate, and expressed as a proportion of the initial wealth, is given by

$$BS(b, \gamma, t) = \frac{e^{-\rho(t-s)}[1 - \alpha^*(t)]X(t)}{x_s} = b^{\frac{1}{1-\gamma}}c^*(s)e^{(r - \beta - \rho + (\mu - r)w^*)(t-s)}e^{\sigma w^* W(t-s) - 0.5\sigma^2 w^{*2}(t-s)} \tag{27}$$

while the corresponding stochastic monetary consumption rate is

$$IS(b,\gamma,t) = \frac{e^{-\rho(t-s)}c^*(b,\gamma,t)X(t)}{x_s} = c^*(s)e^{(r-\beta-\rho+(\mu-r)w^*)(t-s)}e^{\sigma w^* W(t-s)-0.5\sigma^2 w^{*2}(t-s)}. \tag{28}$$

Thus, the bequest amount on death is a fixed multiple, $b^{\frac{1}{1-\gamma}}$, of the monetary consumption rate at all ages. So, an investor who is not sure how to set b can simply be asked what multiple of yearly income he/she would like to bequeath. The expected (discounted) bequest and monetary consumption rates are

$$B(b,\gamma,t) = \frac{e^{-\rho(t-s)}[1-\alpha^*(t)]\mathbb{E}(X(t))}{x_s} = b^{\frac{1}{1-\gamma}}c^*(s)e^{(r-\beta-\rho+(\mu-r)w^*)(t-s)} \tag{29}$$

and

$$I(b,\gamma,t) = \frac{e^{-\rho(t-s)}c^*(b,\gamma,t)\mathbb{E}(X(t))}{x_s} = c^*(s)e^{(r-\beta-\rho+(\mu-r)w^*)(t-s)}. \tag{30}$$

If an investor wishes to aim for a level expectation of discounted income and bequest amounts, we see that both are simultaneously achievable when

$$r - \beta - \rho + (\mu - r)w^* = 0. \tag{31}$$

4. Numerical Results

We use UK Office of National Statistics life tables[3] to calculate $c^*(b,\gamma,t)$, $B(b,\gamma,t)$, and $1-\alpha^*(b,\gamma,t)$ for a male (appendix 1) and female (appendix 2) who start decumulating their pension pot at age $s = 65$ years. We take $\rho = r = 0.02$, $\mu = 0.05$, and $\sigma = 0.2$, all per year. We examine bequest motives in the range $b \in [0,7]$ and constant relative risk aversions in the range $1 - \gamma \in (0,\infty)$. With these values the proportion of current wealth invested in the risky asset is $w^* = \frac{0.05-0.02}{0.2^2(1-\gamma)}$ and so this does exceed 1 when $\gamma > 0.25$. The narrative below is made with respect to appendix 1 only, as the broad pattern is repeated for females in appendix 2.

Figure 1.1: Male, $\gamma = 0.8$

The conspicuous feature here is the potential for huge bequest amounts, albeit with infinitesimally small probabilities, for those living beyond age 95, for every non-zero b. This is possible by borrowing for over-investment in the risky asset, and crucially, by having in the early years, very low consumption proportions coupled with small proportions in the bequest account. This is so the pension pot benefits from large longevity credits in the early years. It is only in the later years, assuming survival, that proportions in the bequest amount are raised significantly. To do otherwise would result in large amounts in the pension pot on death at an advanced age, but most of it going into longevity credits to other members of the tontine.

Figure 1.2: Male, $\gamma = 0.5$

There is still over-investment in the risky asset, that is $w^* = 1.5$, with impressive expected bequest amounts at all ages. In this case the value of b differentiates the expected bequest amounts more so

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than in figure 1.1. Compared with figure 1.1, note the increased consumption proportions and higher proportions in the bequest account, even in the early years.

Figure 1.3: Male, $\gamma = 0.25$

This is still a relatively low level of risk aversion as evidenced by the fact that the pension pot is fully invested in the risky asset ($w^* = 1$). The withdrawal proportions are not that different from those in the case of figure 1.2, while there is the prospect of moderate and increasing bequest amounts at all ages.

Figure 1.4: Male, $\gamma = 0.0001$, the log utility

There is still a large proportion invested in the risky asset ($w^* = 0.75$) which some may think is still indicative of low risk aversion. Compared with figure 1.3 there is generally little difference in the consumption proportions, particularly after the early years, and, as expected with somewhat higher risk aversion, smaller proportions are invested in the bequest account at all ages and for all bequest motives. A notable feature is that the expected bequest amount, although increasing with age, is remarkably stable.

Figure 1.5: Male, $\gamma = \frac{23-3\sqrt{73}}{32} = -0.08225$

This is the solution that simultaneously gives a level expectation of discounted monetary consumption rate and discounted bequest amount, as determined by result (31). These are respectively multiples $c^*(65)$ and $b^{\frac{1}{1.08225}}c^*(65)$ of the pension pot value at age 65.

b	0	0.5	1	2	3	4	5	6	7
Monetary consumption rate p.a. (%)	6.71	6.55	6.41	6.19	5.95	5.75	5.57	5.40	5.24
Bequest amount (%)	0	3.45	6.42	11.7	16.4	20.7	24.7	28.3	31.7

Given a life expectancy at age 65 of approximately 19 years, it can be seen that this appears to be a fair trade-off between bequest and consumption. Indeed, one might almost suggest that delivering level discounted outcomes is the most neutral of all relative risk-averse strategies.

Figure 1.6: Male, $\gamma = -0.5$

In this case one half of the pension pot is invested in the risky asset. With increased risk aversion, expected bequest amounts are smaller than in previous cases. For the first time, the expectations of bequest and monetary consumption rate, both discounted, are decreasing with age. Naturally, the initial monetary consumption rate at age 65 is larger than for the previous cases, which were less risk-averse.

Figure 1.7: Male $\gamma = -2$

Increased risk aversion is responsible for the lower ($w^* = 0.25$) investment in the risky asset and lower bequest amounts. This is accompanied by larger monetary consumption rates at age 65 than in previous scenarios, particularly with higher bequest motives. Thus $c(7,-2,65) = 0.0627$, $c(0,-2,65) = 0.0659$, compared with say $c(7,0,65) = 0.0502$, $c(0,0,65) = 0.0651$. Monetary

consumption rates at age 65 depend less upon the strength of the bequest motive when compared with say figure 1.2, a low risk aversion scenario.

Figure 1.8: Male $\gamma = -6.5$

This represents very high risk aversion, leading to 10% invested in the risky asset. Bequest amounts are small with an emphasis on securing income in the early years at the cost of reductions for long survivors.

Figure 1.9: Male $\gamma = -\infty$

With extreme risk aversion, everything is invested in the riskless asset. At age 65 the monetary consumption rate is the same regardless of the strength of a positive bequest motive, while at large ages it is significantly larger when there is no bequest motive.

Figures 2.1-2.9: Corresponding results for UK females

Figure 3.1 Distribution of discounted wealth at age 95 for female initiating withdrawal at age 65 with $b = 3, \gamma = 0.3, \mu = 0.05, r = \rho = 0.02, \sigma = 0.2$.

With a low constant relative risk aversion of 0.7, $w^* = \frac{15}{14}$ indicating over-investment in the risky asset. Consequently, the distribution is highly skewed with a high probability of running the pension pot to near zero value, even though the expected value at age 95 is 62% of its value at age 65.

Figure 3.2 Distribution of discounted wealth at age 95 for female initiating withdrawal at age 65 with $b = 3, \gamma = -2, \mu = 0.05, r = \rho = 0.02, \sigma = 0.2$.

Here, the risk aversion is much higher at 3. The result is a lower expected value of 16% of initial value, but with a more symmetric distribution, giving in practice some assurance of a small bequest

5. The problem of pure decumulation with a bequest motive

It is relevant to consider the problem of pure decumulation (i.e. no tontine) of a pension pot under a bequest motive, by forcing $\alpha(t)$ to be zero. In that case

$$V(t,x) = \max_{\{w(s),c(s):s>t\}} \mathbb{E}\left(\int_t^\infty \exp\left[-\int_t^s [\lambda(u) + \rho]du\right]\{U[c(s)X(s)] + b\lambda(s)U[X(s)]\}ds \,\Big|\, X(t) = x\right)$$

(32)

subject to

$$\frac{dX(t)}{X(t)} = \left(r + (\mu - r)w(t) - c(t)\right)dt + \sigma w(t)dW(t).$$

(33)

The Hamilton-Jacobi-Bellman approach gives

$$[\lambda(t) + \rho]V(t,x)$$
$$= \max_{w,c}\left\{\frac{(cx)^\gamma}{\gamma} + b\lambda(t)\frac{x^\gamma}{\gamma} + \frac{\partial V(t,x)}{\partial t} + x\frac{\partial V(t,x)}{\partial x}[r + w(\mu - r) - c]\right.$$
$$\left. + \frac{1}{2}x^2\sigma^2w^2\frac{\partial^2 V(t,x)}{\partial x^2}\right\}.$$

$$(34)$$

The first and second order conditions on c and w are as before, as is the expression for $V(x,t)$. Substituting these into (33)

$$\frac{1}{c^*(t)}\frac{dc^*}{dt} = c^*(t) + \frac{b\lambda(t)c^*(t)^{1-\gamma}}{1-\gamma} - \psi(t) \qquad (35)$$

where

$$\psi(t) = k(t) + \frac{\gamma\lambda(t)}{1-\gamma}. \qquad (36)$$

Equation (35) is a Bernoulli differential equation only when $\gamma = 0$ or $b = 0$, so closed-form solutions are possible only in these cases. Stamos (2008) considers decumulation of pooled annuity funds with no bequest motive and has derived, for the degenerate case of only one member in the pool, an identical result to (35-36) for $b = 0$, his equation (32). Similarly, he has derived a closed-form solution, again for $b = 0$, his equation (34), for the special case of a pool of infinite size, which is the same as my tontine solution, $c^*(0,\gamma,t)$, shown in (15). When $\gamma = 0$, equations (35-36) do admit the closed-form solution (21) even when $b \neq 0$. It is exactly the same as the tontine solution, that is results (13) and (14) with $\psi(t) = \lambda(t) + \rho$. Therefore, we conclude that for a logarithmic utility with bequest motive, the optimal fractional consumption for pure decumulation and tontine are the same. Further, subtracting γ^{-1} from the utility function and taking the limit as $\gamma \to 0$ gives the expected lifetime utility as $V(t,x) = \frac{\ln x}{c^*(b,0,t)}$.

6. Summary and Conclusions

In this paper I have obtained closed-form solutions for the optimal fractional consumption rate, bequest-proportion, wealth, and risky/riskless investment ratio for a modern, ideal, explicit tontine with bequest motive, under a constant relative risk aversion utility function. It is shown that the first two are functions of the individual's age, independently of the wealth. The optimal bequest-proportion is a product of the optimal fractional consumption rate and an exponentiated bequest parameter. The derived expressions are simple enough to be computed directly within a life table spreadsheet, making the computations accessible to non-specialists. Some numerical results are given for the UK situation using the Office of National Statistics life tables, separately for males and females.

How should an investor decide upon appropriate values for $1-\gamma$ (risk aversion) and b (strength of bequest motive)? The former could be set using $1 - \gamma = \frac{\mu-r}{w^*\sigma^2}$, where w^* is the desired proportion in the risky asset. Alternatively, if the objective is to achieve level expected discounted monetary consumption rates and bequest amounts, then one would resort to result (31) as in figures 1.5 and 2.5. For parameter values likely to be found in real world situations, condition (4) will be satisfied, and in this case the model leads to a natural interpretation and setting of the bequest parameter through

$$b = \left(\frac{bequest\ amount}{monetary\ consumption\ rate}\right)^{risk\ aversion}.$$

It is also of interest to note that a related problem of how to decumulate a pension pot (no tontine) under a bequest motive, is solvable by suppressing the optimization with respect to the bequest proportion, in favour of forcing it to be 1. In this case, closed-form solutions for the fractional consumption rate and wealth are now available for the following: all bequest motives with a logarithmic utility function; all constant relative risk aversion utilities where there is no bequest motive. With a logarithmic utility, the optimal fractional consumption rate is identical to that obtained for the corresponding tontine.

Bernhardt and Donnelly's (2019) paper breaks new ground in the theory of tontines. It is of practical importance given a growing interest in alternatives to both annuities and pure decumulation of pension pots. Their paper does not allow for the more realistic dynamic optimization of the bequest proportion; perhaps it was thought this would unduly complicate the model and make it less amenable to practical application. In this paper, I have shown that the opposite is in fact the case and that the solution is simplified.

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Appendix 1: Numerical results for Males using UK ONS life tables

Figure 1.1: Male, $\gamma = 0.8$

Withdrawal proportion at different ages, UK male, γ=0.8, w*=3.75

Expected discounted bequest amount as proportion of wealth at age 65, UK male, γ=0.8, w*=3.75

Proportion of wealth, at different ages, to be held in bequest account, UK male, γ=0.8, w*=3.75
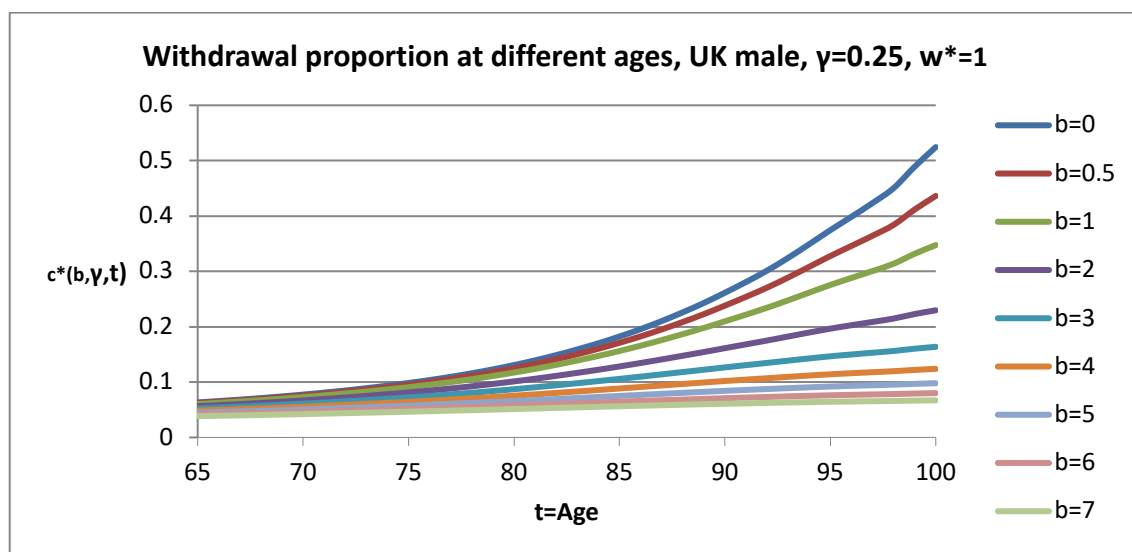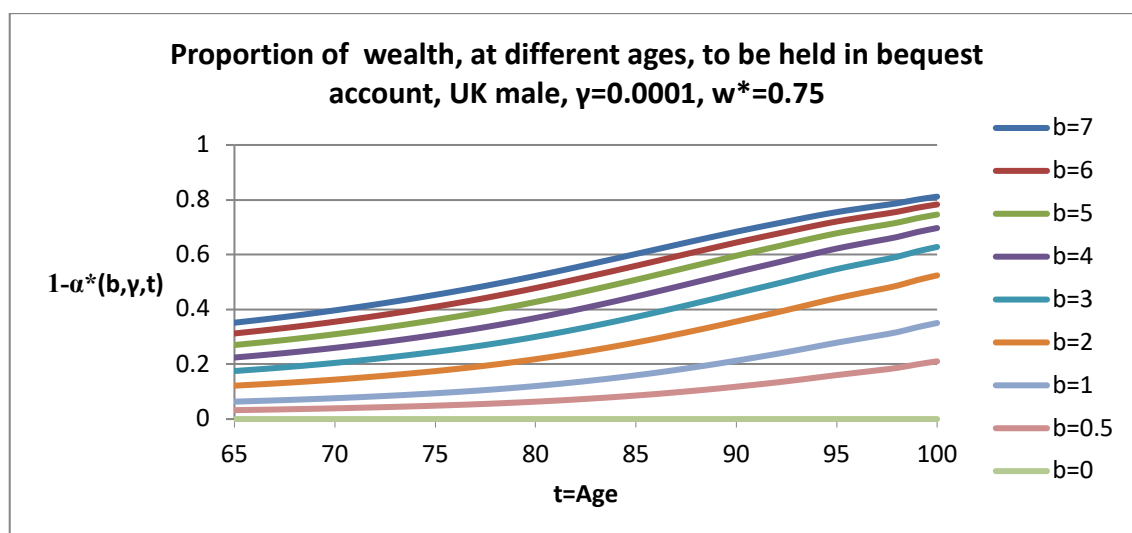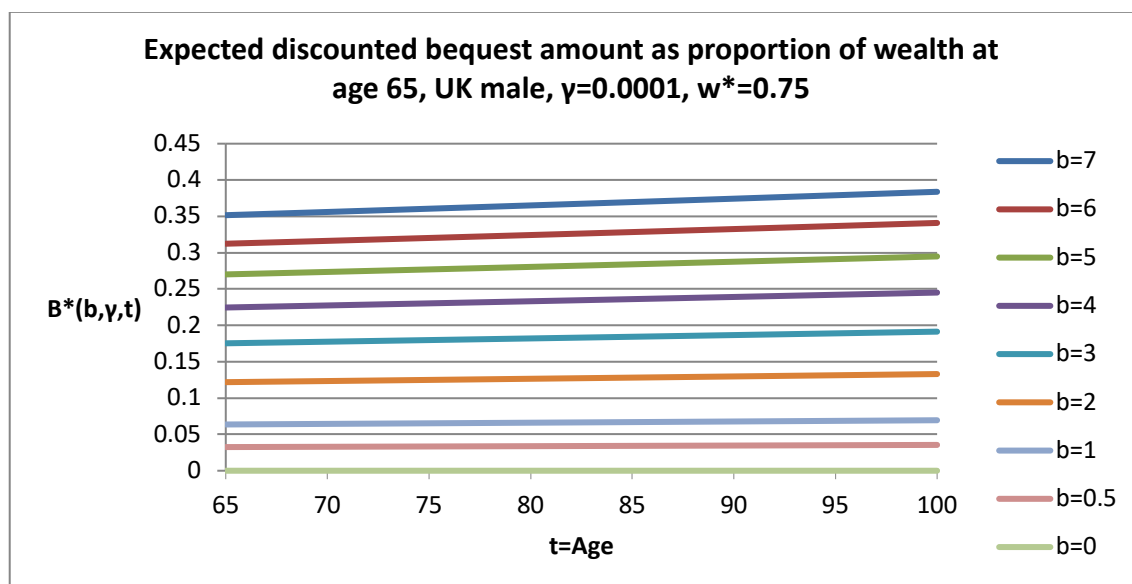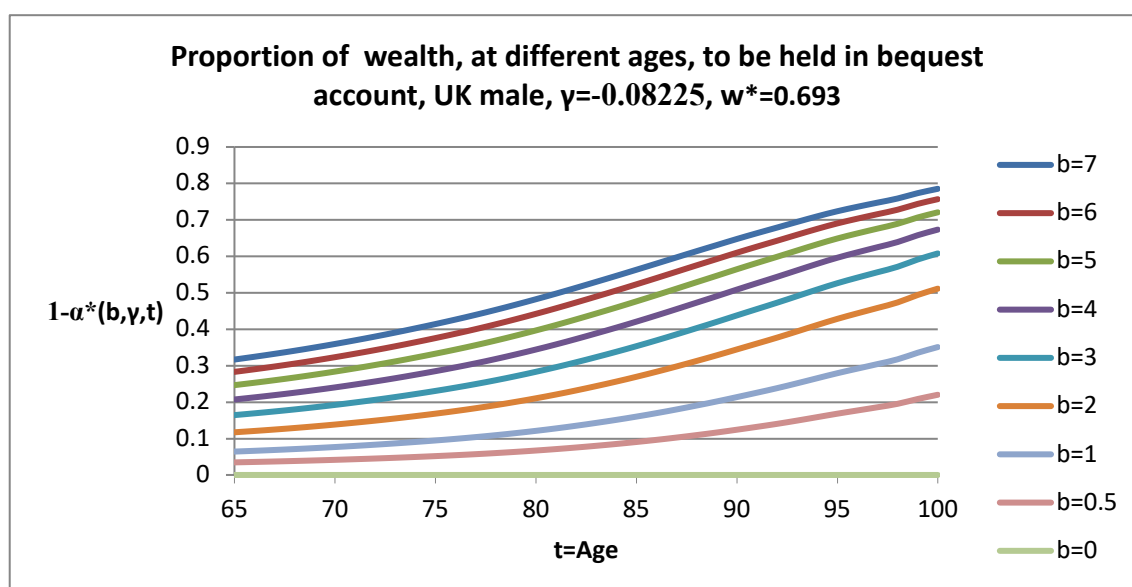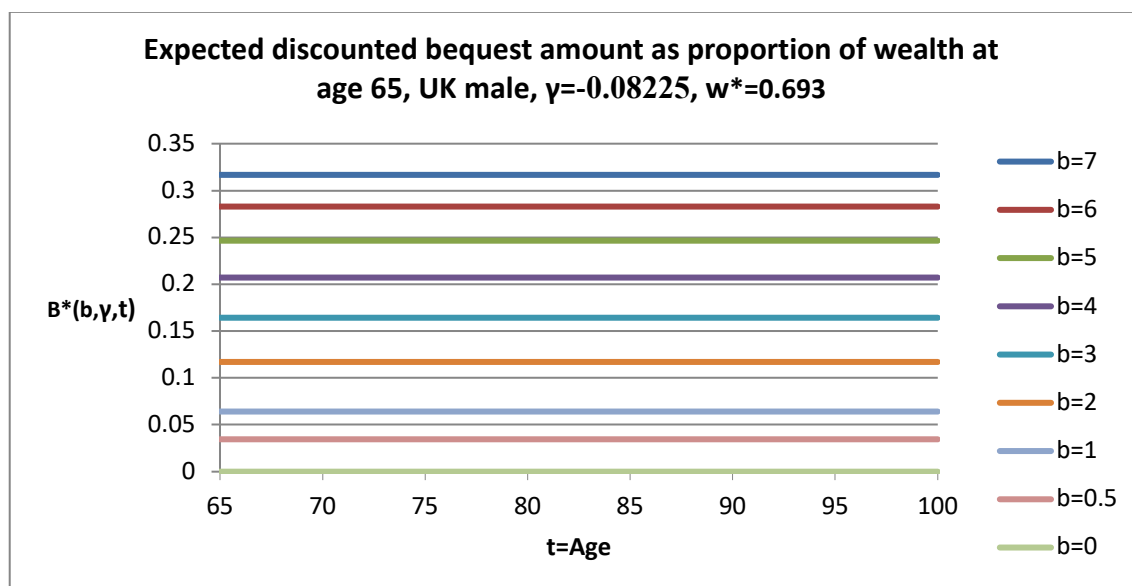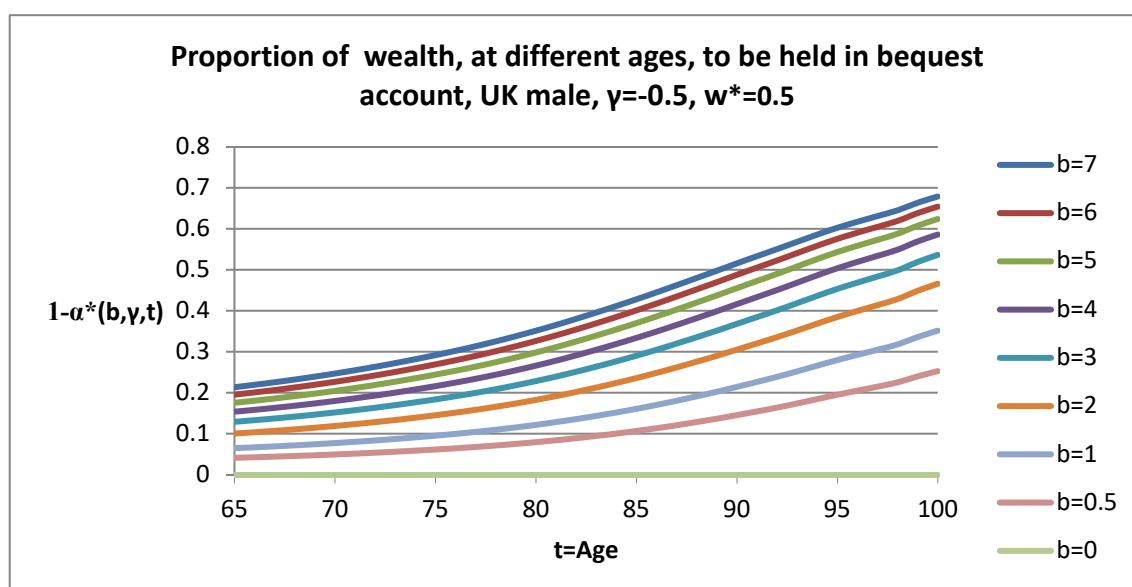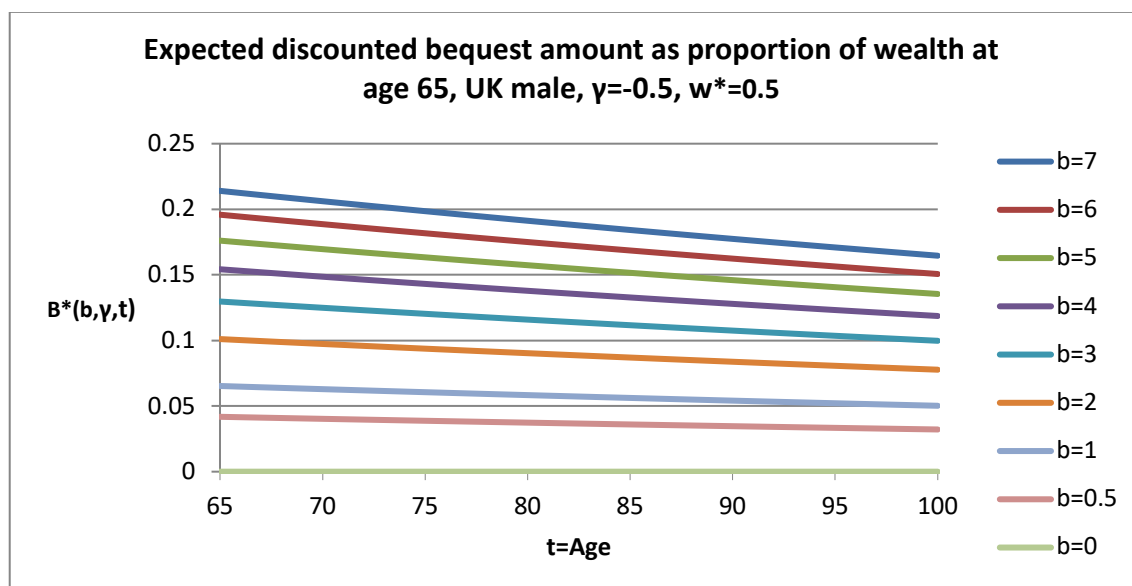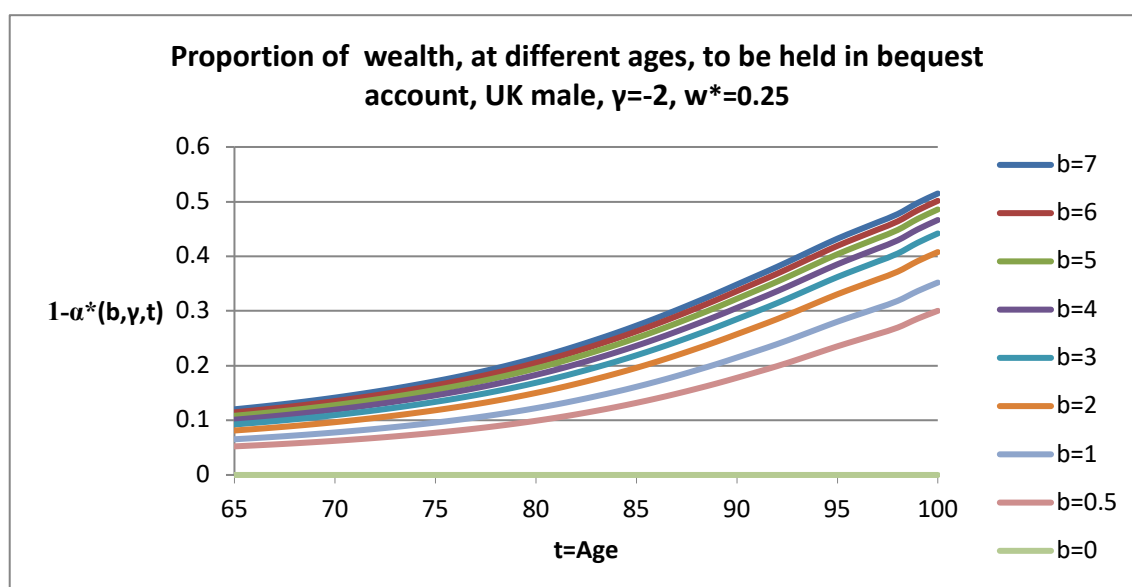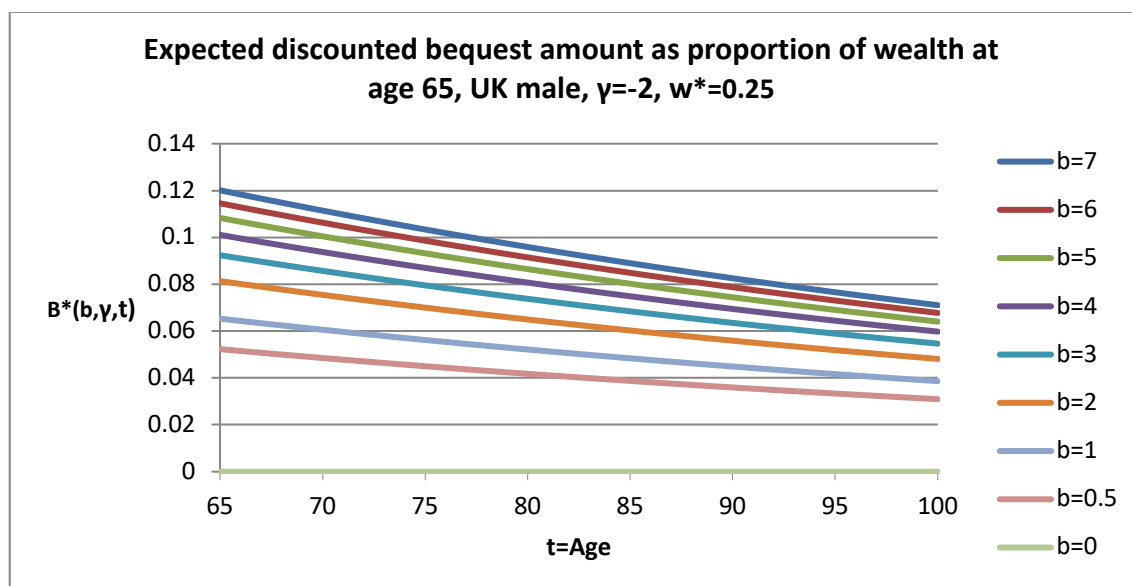
Figure 1.2: Male, $\gamma = 0.5$

Withdrawal proportion at different ages, UK male, γ=0.5, w*=1.5

Expected discounted bequest amount as proportion of wealth at age 65, UK male, γ=0.5, w*=1.5

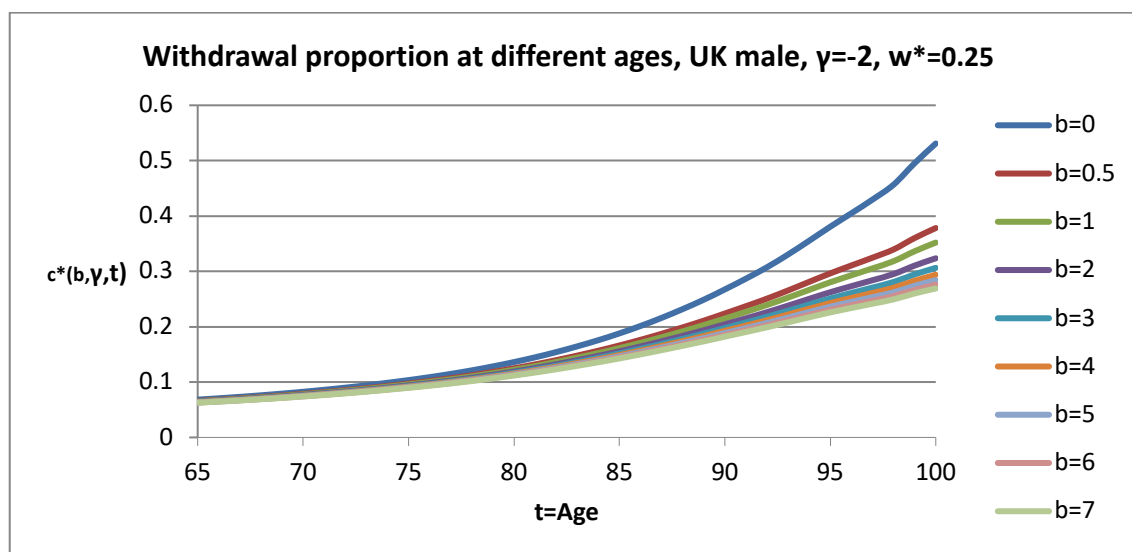Proportion of wealth, at different ages, to be held in bequest account, UK male, γ=0.5, w*=1.5

Figure 1.3: Male, $\gamma = 0.25$

Withdrawal proportion at different ages, UK male, γ=0.25, w*=1

Expected discounted bequest amount as proportion of wealth at age 65, UK male, γ=0.25, w*=1

Proportion of wealth, at different ages, to be held in bequest account, UK male, γ=0.25, w*=1

Figure 1.4: Male, $\gamma = 0.0001$

Withdrawal proportion at different ages, UK male, γ=0.0001, w*=0.75

Expected discounted bequest amount as proportion of wealth at age 65, UK male, γ=0.0001, w*=0.75

Proportion of wealth, at different ages, to be held in bequest account, UK male, γ=0.0001, w*=0.75

Figure 1.5: Male, $\gamma = -0.08225$

Withdrawal proportion at different ages, UK male, γ=-0.08225, w*=0.693

Expected discounted bequest amount as proportion of wealth at age 65, UK male, γ=-0.08225, w*=0.693
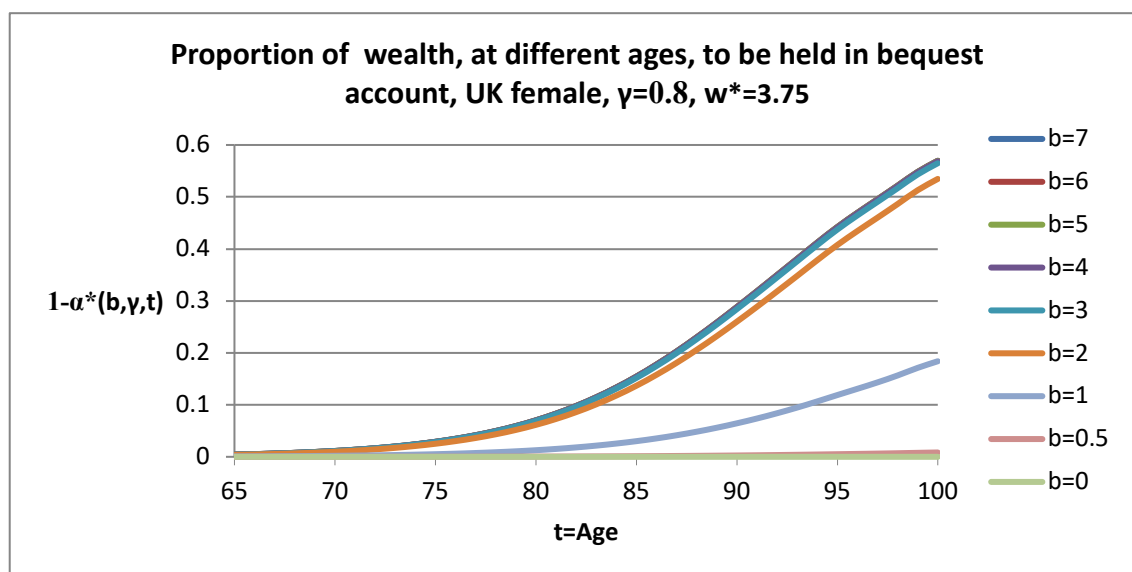

Proportion of wealth, at different ages, to be held in bequest account, UK male, γ=-0.08225, w*=0.693

Figure 1.6: Male, $\gamma = -0.5$

Withdrawal proportion at different ages, UK male, γ=-0.5, w*=0.5

Expected discounted bequest amount as proportion of wealth at age 65, UK male, γ=-0.5, w*=0.5

Proportion of wealth, at different ages, to be held in bequest account, UK male, γ=-0.5, w*=0.5

Figure 1.7: Male, $\gamma = -2$

Withdrawal proportion at different ages, UK male, γ=-2, w*=0.25

Expected discounted bequest amount as proportion of wealth at age 65, UK male, γ=-2, w*=0.25
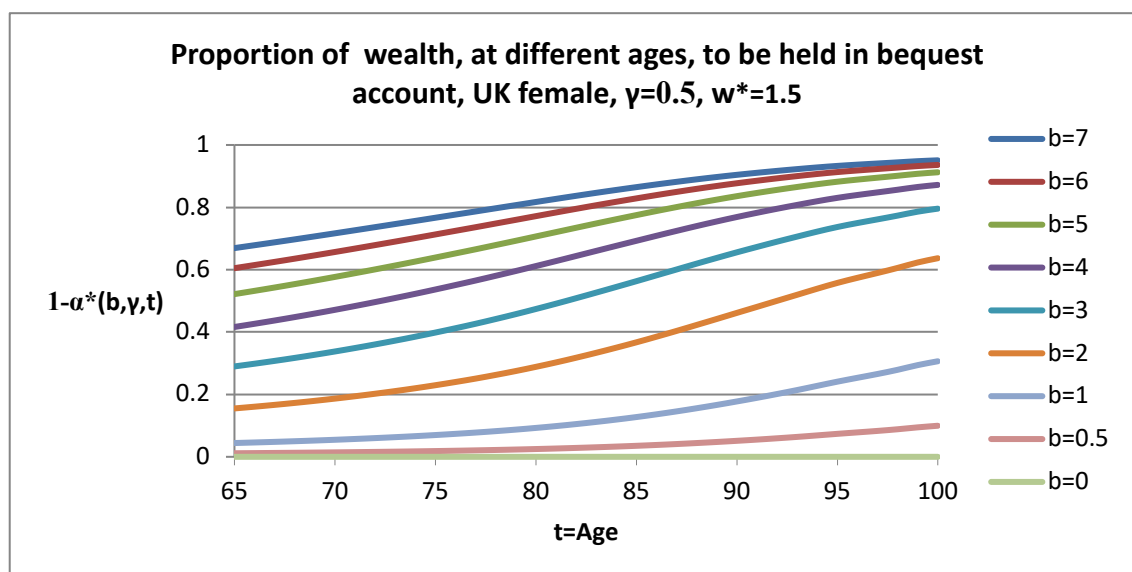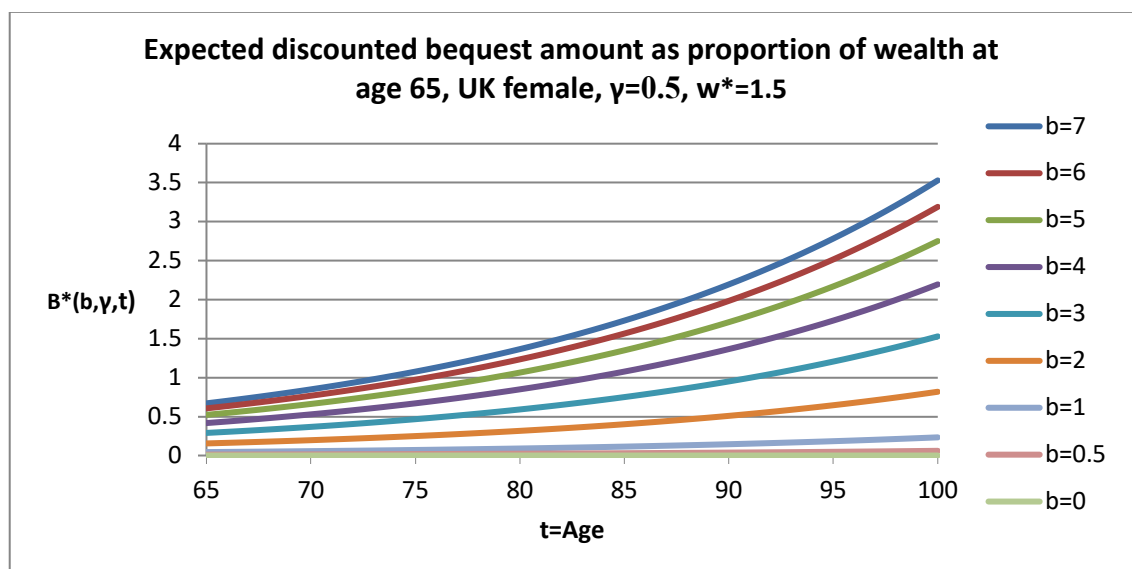
Proportion of wealth, at different ages, to be held in bequest account, UK male, γ=-2, w*=0.25

Figure 1.8: Male, $\gamma = -6.5$

Withdrawal proportion at different ages, UK male, γ=-6.5, w*=0.1

Expected discounted bequest amount as proportion of wealth at age 65, UK male, γ=-6.5, w*=0.1

Proportion of wealth, at different ages, to be held in bequest account, UK male, γ=-6.5, w*=0.1

Figure 1.9: Male, $\gamma = -\infty$

Withdrawal proportion at different ages, UK male, γ=-∞, w*=0

c*(b,γ,t)

Legend: b=0, b=0.5, b=1, b=2, b=3, b=4, b=5, b=6, b=7

t=Age

Expected discounted bequest amount as proportion of wealth at age 65, UK male, γ=-∞, w*=0

B*(b,γ,t)

Legend: b=7, b=6, b=5, b=4, b=3, b=2, b=1, b=0.5, b=0

t=Age

Proportion of wealth, at different ages, to be held in bequest account, UK male, γ=-∞, w*=0

1-α*(b,γ,t)

Legend: b=7, b=6, b=5, b=4, b=3, b=2, b=1, b=0.5, b=0

t=Age

Appendix 2: Numerical Results for Females using UK ONS life tables

Figure 2.1: Female, $\gamma = 0.8$

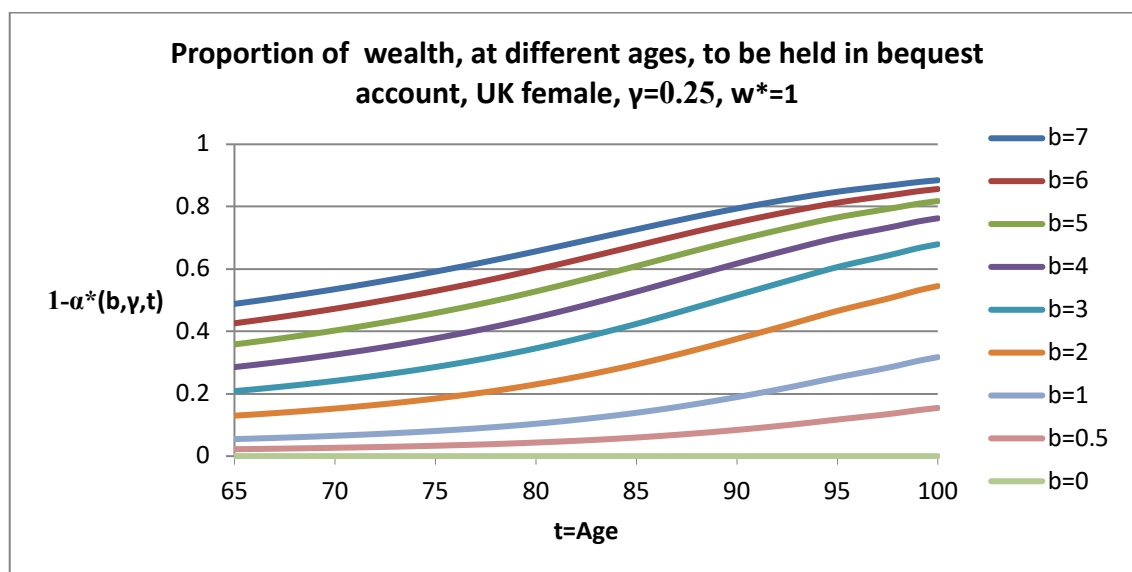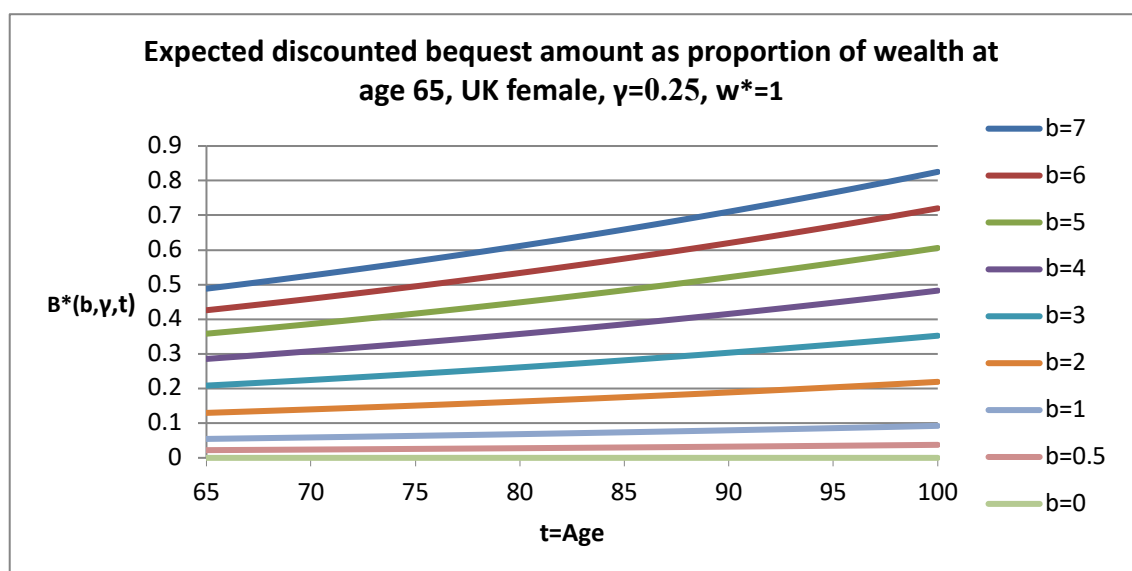**Withdrawal proportion at different ages, UK female, γ=0.8, w*=3.75**

$c^*(b,\gamma,t)$ — t=Age

Legend: b=0, b=0.5, b=1, b=2, b=3, b=4, b=5, b=6, b=7

Expected discounted bequest amount as proportion of wealth at age 65, UK female, γ=0.8, w*=3.75

$B^*(b,\gamma,t)$ — t=Age

Legend: b=7, b=6, b=5, b=4, b=3, b=2, b=1, b=0.5, b=0

Proportion of wealth, at different ages, to be held in bequest account, UK female, γ=0.8, w*=3.75

$1-\alpha^*(b,\gamma,t)$ — t=Age

Legend: b=7, b=6, b=5, b=4, b=3, b=2, b=1, b=0.5, b=0

Figure 2.2: Female, $\gamma = 0.5$

Withdrawal proportion at different ages, UK female, γ=0.5, w*=1.5

c*(b,γ,t) vs *t=Age*

Legend: b=0, b=0.5, b=1, b=2, b=3, b=4, b=5, b=6, b=7

Expected discounted bequest amount as proportion of wealth at age 65, UK female, γ=0.5, w*=1.5

B*(b,γ,t) vs *t=Age*

Legend: b=7, b=6, b=5, b=4, b=3, b=2, b=1, b=0.5, b=0

Proportion of wealth, at different ages, to be held in bequest account, UK female, γ=0.5, w*=1.5

1-α*(b,γ,t) vs *t=Age*

Legend: b=7, b=6, b=5, b=4, b=3, b=2, b=1, b=0.5, b=0

Figure 2.3: Female, $\gamma = 0.25$

Withdrawal proportion at different ages, UK female, γ=0.25, w*=1

Expected discounted bequest amount as proportion of wealth at age 65, UK female, γ=0.25, w*=1

Proportion of wealth, at different ages, to be held in bequest account, UK female, γ=0.25, w*=1

Figure 2.4: Female, $\gamma = 0.0001$

Withdrawal proportion at different ages, UK female, γ=0.0001, w*=0.75

Expected discounted bequest amount as proportion of wealth at age 65, UK female, γ=0.0001, w*=0.75

Proportion of wealth, at different ages, to be held in bequest account, UK female, γ=0.0001, w*=0.75

Figure 2.5: Female, $\gamma - 0.08225$

Withdrawal proportion at different ages, UK female, γ=-0.08225, w*=0.693

Expected discounted bequest amount as proportion of wealth at age 65, UK female, γ=-0.08225, w*=0.693

Proportion of wealth, at different ages, to be held in bequest account, UK female, γ=-0.08225, w*=0.693

Figure 2.6: Female, $\gamma - 0.5$

Withdrawal proportion at different ages, UK female, $\gamma=-0.5$, $w*=0.5$

Expected discounted bequest amount as proportion of wealth at age 65, UK female, $\gamma=-0.5$, $w*=0.5$

Proportion of wealth, at different ages, to be held in bequest account, UK female, $\gamma=-0.5$, $w*=0.5$

Figure 2.7: Female, $\gamma = -2$

Withdrawal proportion at different ages, UK female, γ=-2, w*=0.25

Expected discounted bequest amount as proportion of wealth at age 65, UK female, γ=-2, w*=0.25

Proportion of wealth, at different ages, to be held in bequest account, UK female, γ=-2, w*=0.25
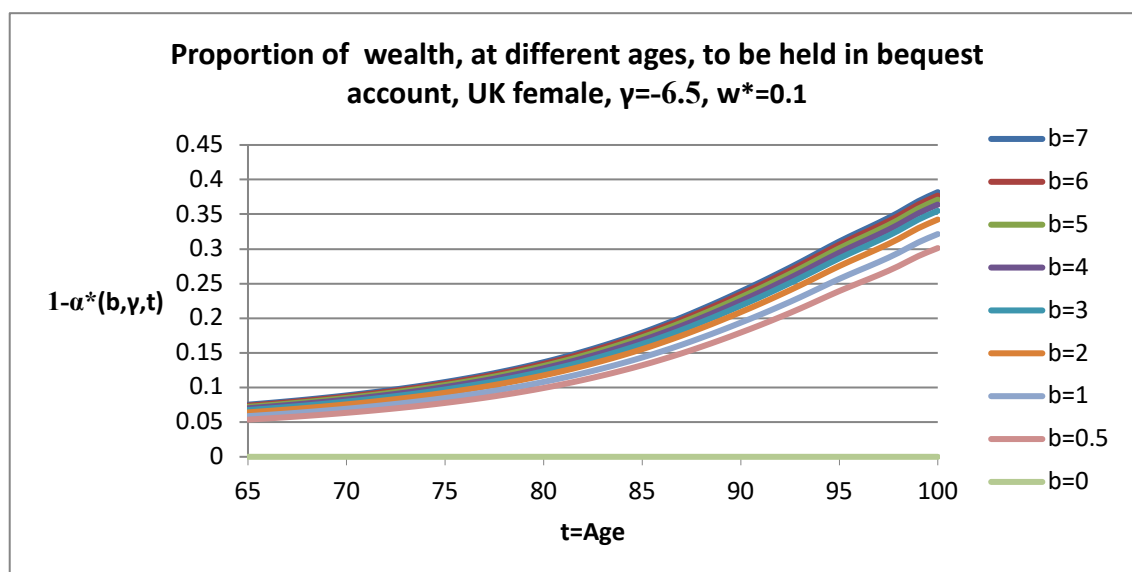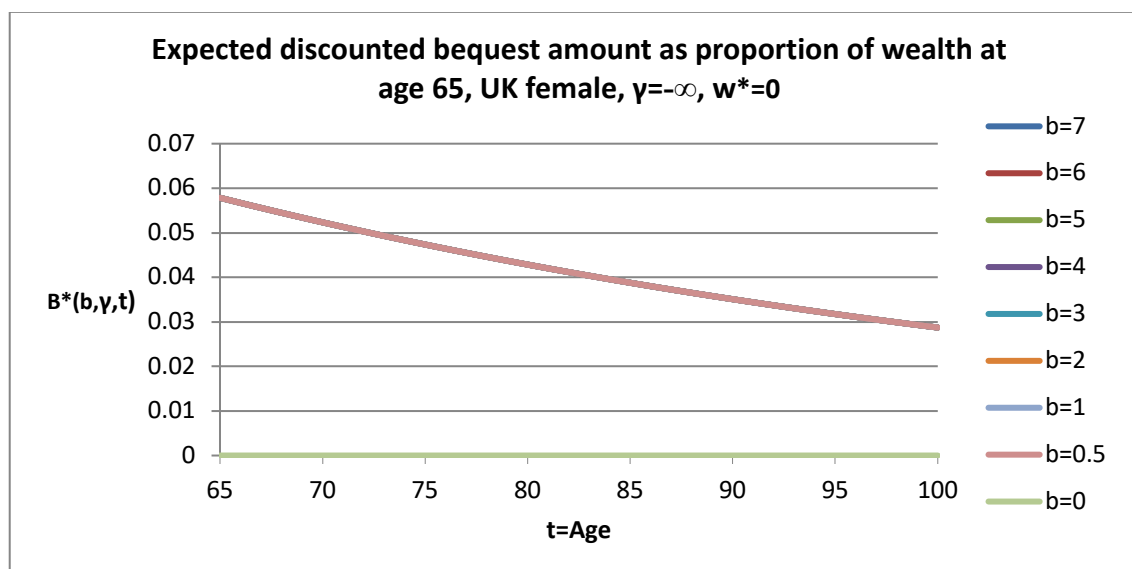
Figure 2.8: Female, $\gamma = -6.5$

Figure 2.9: Female, $\gamma = -\infty$

Withdrawal proportion at different ages, UK female, γ=-∞, w*=0

Expected discounted bequest amount as proportion of wealth at age 65, UK female, γ=-∞, w*=0

Proportion of wealth, at different ages, to be held in bequest account, UK female, γ=-∞, w*=0

Appendix 3: Examples of the Distribution of Discounted Wealth at age 95 for Females

Figure 3.1. Distribution of discounted wealth at age 95 for female initiating withdrawal at age 65 with b=3,γ=0.3,μ=0.05, r=ρ=0.02,σ=0.2

The expected value of discounted pension pot at age 95 is 0.6205, median =0.31.

Figure 3.2. Distribution of discounted wealth at age 95 for female initiating withdrawal at age 65 with b=3, γ=-2, μ=0.05, r=ρ=0.02, σ=0.2

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The expected value of discounted pension pot at age 95 is 0.1602, median =0.155.