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Mortality Regimes and Pricing

Abstract

Mortality dynamics are characterized by changes in mortality regimes. This paper describes a Markov regime switching model which incorporates mortality state switches into mortality dynamics. Using the 1901-2005 US population mortality data, we illustrate that regime switching models can perform better than well-known models in the literature. Furthermore, we extend the Lee-Carter (1992) model in such a way that the time-series common risk factor to all cohorts has distinct mortality regimes with different means and volatilities. Finally, we show how to price mortality securities with this model.

Keywords: Lee-Cater model, regime switching mortality model, mortality-linked securities

JEL classification: C02, C13, G22, G23

1. Introduction

Various exogenous factors influence mortality rates over time. We usually think that severe, short-termed events such as pandemics, food contamination, and tsunamis underlie mortality risk. Mortality risk is the risk of having more deaths than expected, which may imply higher death probabilities across many age cohorts. On the other hand, longevity risk is the risk of more lives surviving than expected or observed death rates being lower than expected. Usually longevity risk is realized over a long period. Various medical advances and health technologies tend to reduce the number of deaths. Moreover, political transformations can create incentives for population mobility, which could introduce (positive or negative) jumps in mortality with lasting effects. These factors could affect mortality for specific age groups or the whole population.

The future mortality process is affected by both mortality risk and longevity risk. In modeling future mortality or longevity risk, the literature considers different scenarios and uses various stochastic processes to model mortality and/or longevity risk (e.g. Milevsky and Promislow 2001; Dahl 2004; Ballotta and Haberman 2006; Cairns et al. 2006; Cox et al. 2006; Dahl and Møller 2006; Gründl et al. 2006; Lin and Cox 2008; Kogure and Kurachi 2010; Wills and Sherris 2010; Yang et al. 2010). What distinguishes the approach in this paper and in the literature on mortality risk modeling is that it explicitly delineates the pattern of changes in mortality rates in various states via regime switching techniques.

Regime switching (RS) models were introduced by Goldfeld and Quandt (1973) and later improved by Hamilton (1989) who used RS models to describe structural changes in economic nonstationary time series data. Hamilton and Susmel (1994) extend the model into multiple regimes with autoregressive coefficients. Since introduced, various RS models have been used in the actuarial and insurance literature to capture non-normality characteristics present in asset prices (Hardy 2001), asset allocation (Boyle and Liew 2007), options (Naik 1993; Bollen 1998; Boyle and Draviam 2007; among others), credit risk (Siu, Erlwein and Mamon 2008) equity prices in response to information about changes in credit ratings (Milidonis and Wang 2007), pricing investment guarantees (Lin, Tan and Yang 2009; Boudreault and Panneton 2009) among other topics of interest. .

We argue that RS models can be applied to mortality risk modeling: RS models have some desirable features that allow them to capture some major characteristics of the current mortality universe. First, RS models can describe mortality changes through different means and volatilities in the various switching states. Also, they can identify the time at which a shock arrives to the underlying mortality variable. These properties allow for a more flexible model which can accommodate jumps and changes in volatility in the mortality index.

Second, RS models can reflect different natures of mortality evolutions: either a temporary mortality shock or permanent mortality improvement. The annual conditional probability of being in each implied regime provides an attractive feature as it essentially provides a probabilistic propagation of the mortality trend among the assigned regimes. Transition probabilities between different regimes p_{ij} (regimes *i*, *j*) are the parameter estimates resulting from the conditional regime probability. The higher p_{ij}, the faster the shock effect will die out. Therefore, RS models allow the user to observe the deterioration of mortality jumps or a longer-lasting longevity effect and provide the user a more in-depth view of the trend underlying mortality. Many mortality jump models do not have this valuable feature. Furthermore, depending on anticipated changes in mortality risk associated with an insurance company's portfolio, RS parameters can be altered to include higher/lower jumps, longer/ shorter duration of regimes and higher/lower volatility per regime.

Third, RS models can accommodate non-normality features, such as multi-modality, skewness and excess kurtosis, in an underlying loss distribution. Existing models (e.g. Lee-Carter 1992) usually assume normality in the error term of the common time-series risk factor that affects all age cohorts across years. However, the normality assumption may not hold in some cases.

Given the attractive properties and flexibility of RS models, this paper proposes how RS models can be constructed and used to describe the dynamics of population mortality indices. At first we show that RS models can perform better than the existing models in the literature, using the 1901-2005 US population mortality index as an example. The second application relates to the error term from the time-series common risk factor to all cohorts in the Lee-Carter (1992) model. We extend the Lee-Carter (1992) model by allowing the time-series common risk factor to all cohorts to move between regimes. To our knowledge, this paper is the first in the actuarial science literature to apply RS models to model mortality dynamics. Finally, we show how to forecast and use the forecasted mortality rates obtained from the RS technique for longevity security pricing.

2. General Form of RS Estimation without Distribution Restriction

Future mortality is affected by both mortality and longevity shocks. RS models are flexible enough to accommodate a range of distributions on the underlying mortality hidden states. Non-normal observations in the underlying mortality index distribution may be introduced by sudden mortality spikes due to catastrophe death events or excess mortality improvements over an extended period through the advancement of medical technology or the adoption of healthier lifestyles. The beauty of RS models is that any (e.g. short-lived) structural changes that might appear in the evolution of mortality indices are isolated in a new (e.g. short-lived) regime. Moreover, the model has the flexibility to choose through the maximum likelihood, the switching time, the duration as well as the value of the parameter estimates. Finally the average duration of each regime can be easily estimated from the probability transition matrix of the Markov chain. All these attractive properties of RS models provide us a fresh look at the dynamics of mortality indices.

As the first step, we present the general case of the two-state RS model without specifying the underlying loss distribution at each state. In Section 3 and Section 4, we state our distributional and parametric assumptions that customize a RS model to each of the two time-series datasets analyzed in this paper.

Suppose that we have an annual index S_t, whose log change rate for year t is Y_t:

$$Y_t = \log\left(S_t / S_{t-1}\right). \tag{2.1}$$

Assume Y_t is governed by an unobserved Markov process with two regimes: $\rho_t = 1$ or 2. Each regime is represented by a loss distribution, ${}^{\rho_t}f\left(y\right)$. More specifically, the process can be summarized by:

$$Y_t = \begin{cases} {}^1Y_t & if \ \rho_t = 1 \\ {}^2Y_t & if \ \rho_t = 2 \end{cases}. \tag{2.2}$$

In equation (2.2), ${}^{\rho_t}Y_t$ characterizes the process in regime ρ_t between time periods $[t, t+1)$. If Y_t's are conditionally independent, the Markovian probability transition matrix that describes the random switching between the two regimes is:

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \tag{2.3}$$

where $p_{jk} = \Pr[\rho_{t+1} = k \mid \rho_t = j]$ is the probability that the annual log change rate process will switch from regime j at time t to regime k at time $t+1$, where ($j = 1,2$ and $k = 1,2$). We let Θ represent the set of parameters of the RS model, which is estimated using numerical optimization techniques to maximize the log-likelihood function. The log-likelihood function of the RS model is defined by

$$Log[L(\Theta)] = \sum_{t=1}^{n} \log[f(y_t \mid \Theta, y_1, y_2, y_3, \ldots y_{t-1})] \tag{2.4}$$

where f is the probability density function of y and n is the total number of annual log change rates. In Appendix A we explain the recursive estimation of RS models where numerical optimization is employed to estimate Θ.

3. RS-GBM model for US population mortality index

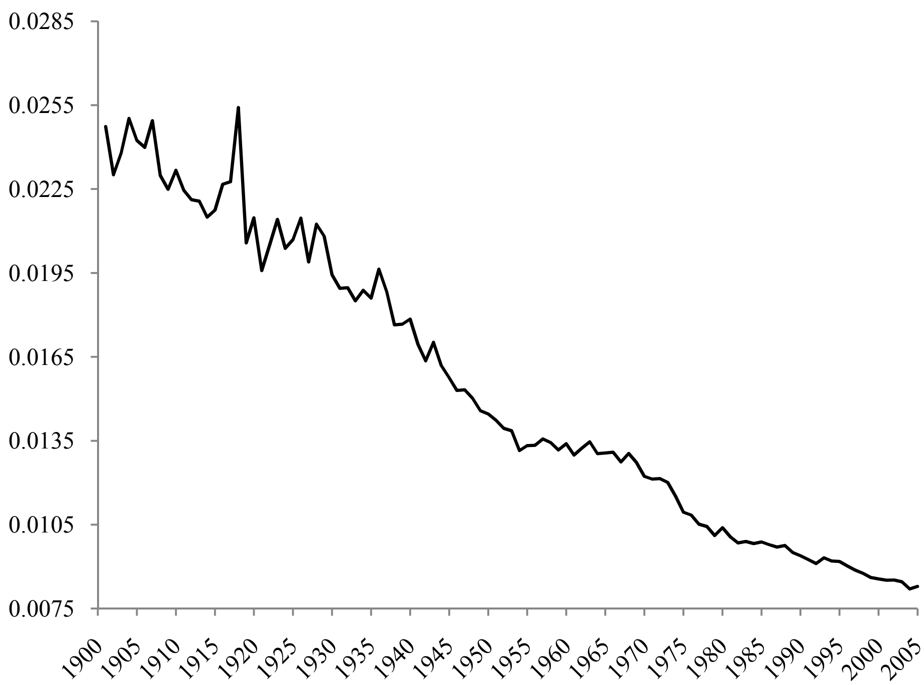
To apply RS models in mortality risk modeling, we show how to describe the US population mortality index with a RS model between two geometric Brownian motions (RS-GBM). The RS-GBM model allows jumps to take place in any direction and we do not impose any constraints on the value of our parameters.

3.1 Data

The data for our estimation come from two databases: the 1901-1999 US population mortality tables are downloaded from the Human Life Table Database[1] and the 2000-2005 tables are from the Human Mortality Database[2]. Specifically, the age range is $x = 0, 1, 2, \ldots, 100$ for the US population and the time period is $t = 0, 1, 2, \ldots, 104$ representing calendar years 1901, 1902, 1903,..., 2005. They provide a total of 10,605 cells. To calculate the US population mortality index or population death rate in year t, q_t, we first sum up the total number of deaths across different ages of the US population (including both males and females) during that year and then divide it by the population size at the beginning of year t. Accordingly, we obtain a time series of q_t where $t = 0, 1, 2, \ldots, 104$, which is illustrated in Figure 1 over time.

Figure 1

US population mortality index from 1901 to 2005



[1] Data source: Human Life Table Database. Max Planck Institute for Demographic Research (Germany), University of California, Berkeley (USA) and the Institut national détudes démographiques (France). Available at www.lifetable.de (data downloaded on June 8, 2008).

[2] Data source: Human Mortality Database. University of California, Berkeley (USA), and Max Planck Institute for Demographic Research (Germany). Available at www.mortality.org or www.humanmortality.de (data downloaded on June 8, 2008).

The *y*-axis in Figure 1 represents the US population mortality index q_t, and the *x*-axis is the year from 1901 to 2005, denoted $t = 0, 1, 2, \ldots, 104$. We observe a downward trend, implying the US population mortality, in general, improved overtime. Meanwhile, during our observation period, there were variations around this trend, reflecting different natures of mortality shocks, either permanent or temporary. For some periods such as the period between 1935 and 1950, the mortality improved at a much higher rate (i.e. a steeper slope) with a long duration but a low volatility. However, in a few periods, notably in the period around 1918 (because of the 1918 worldwide flu), the mortality deteriorated dramatically but temporarily and had a much higher volatility than most of the periods. Such similar changes in various mortality regimes in the future will affect the whole population mortality differently and have important implications for mortality modeling and forecasting.[3]

3.2 RS-GBM model for US Population Mortality Index

The RS-GBM model follows the general model in Section 2, with the following specifications: we replace the annual index S_t in equation (2.1) with the US population mortality index q_t. Hence, the mortality log change rate, $Y_t = \log\left(q_t / q_{t-1}\right)$. Table 1 presents the descriptive statistics of Y_t based on the US population mortality index q_t.

Table 1
Descriptive Statistics of the U.S. population $Y_t = \log\left(q_t / q_{t-1}\right)$

Descriptive Statistics	$Y_t = \log\left(q_t / q_{t-1}\right)$
Mean	-0.0106
Standard error	0.0037
Median	-0.0102
Standard deviation	0.0378
Sample variance	0.0014
Kurtosis	7.4885
Skewness	-1.1321
Range	0.3219
Minimum	-0.2114

[3]In Figure 1, the mortality trend based on the RS-GBM model represents the population mortality evolution over all age groups but the same technique can be applied to a different age group of interest. For instance, in pricing defined benefit plans and annuities, researchers and professionals focus on older age groups. The RS-GBM model can easily fit these groups' mortality experience if data are available. We thank the referee for this insight.

Maximum	0.1104
Number of observations	105

Table 1 highlights the non-normality feature of Y_t, illustrated by the large kurtosis (7.4885) and negative skewness (-1.1321). A RS-GBM model can nicely fit to this situation. Specifically, at each Markov state for the log change rate process Y_t, we attach a normal distribution such that ${}^1Y_t = \mu_1 t + \sigma_1 W_t$ and ${}^2Y_t = \mu_2 t + \sigma_2 W_t$. We call this the model "RS-GBM". This implies the mortality index switches between two geometric Brownian motions, which are characterized by a different mean and volatility that allow log change rates of the mortality index to deviate away from the normality assumption and thus potentially attain fatter tails, higher kurtosis and bi-modality. In the case of normality, our model would simplify to the case of one regime. In summary, even though our model can accommodate non-normality features, at the same time it can be reduced down to a single geometric Brownian motion in the case that Y_t falls within the boundaries of normality.

3.3 Lin and Cox (2008)'s Model for US Population Mortality Index

As part of our estimation we compare RS-GBM to other mortality models such as Lin and Cox (2008)'s model. Lin and Cox (2008) propose an approach to model the dynamics of the US population mortality index. Their method combines a geometric Brownian motion (GBM) and a discrete Markov chain. The GBM describes the US population mortality index $\overline{q}_t$ if no one-time catastrophe death event (e.g. the 1918 worldwide flu) occurs,

$$\mathrm{d}\overline{q}_t = \alpha \overline{q}_t \mathrm{d}t + \sigma \overline{q}_t \mathrm{d}W_t \,, \tag{3.1}$$

where α is the drift of $\overline{q}_t$ and σ is the instantaneous volatility given no jumps. W_t is a standard Brownian motion with mean 0 and variance *t*. The temporary mortality shocks are described by a discrete Markov chain which counts the number of jumps N_t up to year *t* = 0, 1, 2, . . . , 104. Given $N_0 = 0$, the transition

$$N_{t+1} = \begin{cases} N_t + 1 & \text{with probability} \quad p \\ N_t & \text{with probability} \quad 1\text{-}p \end{cases} . \tag{3.2}$$

The probability of having a mortality jump during the period [t, t+1] is p. If a jump occurs in [t, t+1], the percentage change in the mortality index due to the jump is denoted by $R_t - 1$. Lin and Cox (2008) assume R_t is log-normally distributed with parameters r_1 and r_2,

$$R_t = e^{r_1 + r_2 V_t},$$

where V_t is a standard normal variable. The process of the US mortality index q_t is a combination of equations (3.1) and (3.2),

$$q_t = \begin{cases} \bar{q}_t R_t & \text{with probability } p \\ \bar{q}_t & \text{with probability } 1\text{-}p \end{cases} . \tag{3.3}$$

If no jump occurs during [t, t+1], $R_t = 1$. So $q_t = \bar{q}_t$. Otherwise, there is a temporary jump effect of R_t on $\bar{q}_t$, leading to $q_t = \bar{q}_t R_t$.

3.4 Results

Based on the US population mortality index q_t from 1901 to 2005, Table 2 reports our maximum likelihood estimation results for the GBM-only model without the jump process as equation (3.1), Lin and Cox (2008)'s model, and RS-GBM model.

Starting with the GBM-only model, we observe that the drift α and volatility σ are in line with the descriptive statistics of the log change rate Y_t in Table 1. From Lin and Cox (2008)'s model we obtain $\alpha =$ -0.0094. The negative sign of α implies the improvement of US population mortality during this period. The instantaneous volatility of the mortality index, conditional on no jumps, σ, equals 0.0299. The probability of a jump event each year is equal to 0.0121, so we expect about one jump per one hundred years. The log-likelihood value of the model equals 207.15, which is higher than that of the GMB-only model, 195.48. It means Lin and Cox (2008)'s model has a better fit of data than GMB-only.

Table 2
Maximum Likelihood Parameter Estimates of Competing Models Based on the Annual US Population Mortality Index from 1901 to 2005

Statistics	GBM-only	Lin & Cox (2008)	RS-GBM
Log-likelihood	195.48	207.15	219.02
α	-0.0106	-0.0094	
σ	0.0376	0.0299	
p		0.0121	
r_1	o0.1395	0.1395	
r_2		0.0477	
μ_1			-0.0101
μ_2			-0.0109
σ_1			0.0540
σ_2			0.0183
p_{12}			0.0129
p_{21}			0.0105

The RS-GBM model catches the improvement in mortality over time (negative means for both regimes) while all the non-normal deviation in the US mortality log change rates is captured through two distinct regimes of varying volatility $\left(\sigma_1 = 0.0540,\ \sigma_2 = 0.0183\right)$ which are almost equally persistent $\left(\pi_1 = 0.449,\ \pi_2 = 0.551\right)$.[4] The lower log-likelihood value 207.15 of Lin and Cox (2008)'s model implies the RS-GBM model (log-likelihood = 219.02) provides a better fit for the data.[5]

[4] The initial values for our regime switching model estimation are based on the parameters estimates of the GBM-only model. Specifically, $\mu_1 = \mu_2 = \alpha = -0.0106,\ \sigma_1 = 2\sigma = 2\times 0.0376 = 0.0752,\ \sigma_2 = 0.5\sigma = 0.5\times 0.0376 = 0.0188,$ p_{12} = p_{12}=0.5. We also try many other initial values for these six parameters of the RS-GBM model to make sure that the log-likelihood is maximized.

[5] Although Lin and Cox (2008)'s model and the RS-GBM model are not embedded, the log-likelihood values and likelihood ratio test can still be used for model selection even if the Chi-square distribution in this case is only an approximation (Hardy, 2001).

Our likelihood-ratio (LR) test in Table 3 confirms the superiority of the RS-GBM model over the other two models at the significance level of 0.1%. The Schwartz-Bayes and Akaike Information Criteria also show support for the RS-GBM model.[6]

Table 3
Goodness of Fit Tests of Competing Models

Panel A - Goodness of Fit between GBM and RS-GBM

	Akaike IC	Schwartz Bayes IC	LL*
GBM	193.48	190.83	195.48
RS-GBM	213.02	205.06	219.02
LR Test**			47.08
p-value			<0.001
Better Model	RS-GBM	RS-GBM	RS-GBM

Panel B - Goodness of Fit between Cox and Lin (2008) and RS-GBM

	Akaike IC	Schwartz Bayes	LL*
Cox & Lin (2008)	202.15	195.51	207.15
RS-GBM	213.02	205.06	219.02
LR Test**			23.74
p-value			<0.001
Better Model	RS-GBM	RS-GBM	RS-GBM

* Log-likelihood value; ** Likelihood-ratio test.

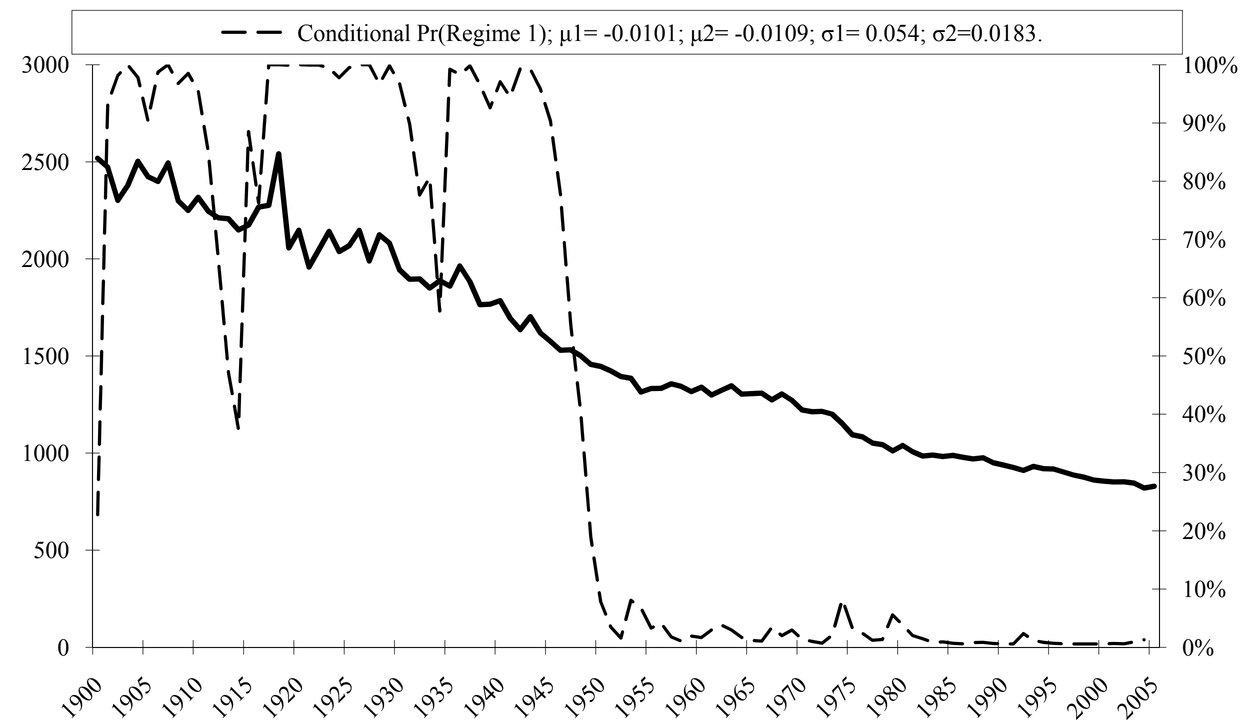
One additional demonstration that lends support to the RS-GBM model is the probabilistic classification of the U.S. mortality index between the two regimes over time, as that is shown in Figure 2. The left *y*-axis shows the number of deaths per 100,000 U.S. population lives each year from 1901 to 2005 (*x*-axis). The right *y*-axis represents the magnitude of the annual conditional probability of being classified in the high volatility regime. For example if a process is classified in regime 1 which is the high volatility regime with $\mu_1 = -0.0101$, $\sigma_1 = 0.054$ (such as the 1918 influenza), its probability will be very close to 1. Other observations in the low volatility regime (e.g. post Second World War period) score about 0% on the right *y*-axis. The two distinct

[6] We also tried other regime switching variations. For example, we constructed a regime switching model between two beta distributions (RS-beta). But its goodness of fit was not better than that of RS-GBM in Table 3. To conserve space, the results based on RS-beta are not shown in the paper.

regimes are evident in Figure 2 along with their parameter estimates. The volatility in mortality rates plays a large role in separating the two regimes. For example in the case of Lin and Cox (2008)`s model r_1 effectively captures large jumps (for example in 1918), however any volatility in mortality is averaged out by the parameter σ . Furthermore, in the case of the GBM-only model, any changes in the mean and volatility are captured by the respective drift and volatility parameters, thus disallowing for non-normality features in log-mortality rates.

Figure 2

Conditional Probability of US Population Mortality Index Classified in High Volatility Regime



3.5 Is Modeling Changes in Mortality Regimes Important?

To prove that modeling changes in mortality regimes with the RS-GBM method is important in pricing mortality securities, we compare the market price of risk based on the RS-GBM model and Lin and Cox (2008)'s model.

Lin and Cox (2008) price the mortality securities with their proposed mortality stochastic model (see equation (2.18)) and the Wang transform. Wang (1996, 2000, 2001) develops an incomplete market pricing method that integrates financial and insurance pricing theories to value property catastrophe bonds. Specifically, Wang defines a probability distortion operator, now known as the Wang transform, as follows:

$$F_\lambda^*(x) = G[\Phi^{-1}(F(x)) - \lambda],$$

where $F(x)$ is the cumulative distribution function of a given asset X and $\Phi(\cdot)$ is the standard normal cumulative distribution function. In Wang (2004), the function G is the Student's *t*-distribution with six degrees of freedom to account for the model uncertainty. The parameter λ reflects the level of systematic risk and measures the market price of risk.

As an example, we now show how the 2005 Swiss Re Vita II mortality bond price differs based on the RS-GBM model and Lin and Cox (2008)'s model. Swiss Re issued its second life catastrophe bond and obtained $362 million of mortality risk coverage through the Vita Capital II in 2005 (Swiss Re 2005). Vita II is a five-year bond with three tranches: Class B, Class C and Class D. The expected maturity is in 2010. The mortality risk is defined in terms of a combined mortality index (CMI), q_t, which is a weighted average of annual population death rates in the United States (62.5%), United Kingdom (17.5%), Germany (7.5%), Japan (7.5%) and Canada (5%). Specifically, the payments of different tranches in year *t* are determined by the value of an index, i_t, in year *t*(t=2006, 2007,..., 2010):

$$i_t = \frac{(q_t + q_{t-1})/2}{(q_{2003} + q_{2002})/2},$$

where $(q_t + q_{t-1})/2$ is the actual average mortality experience in years *t* and *t*-1, and $(q_{2003} + q_{2002})/2$ is the base level equal to the average of the 2002 and 2003 CMIs. The investors will have a reduced principal in year *t* if, during a measurement period of any two consecutive years within the risk coverage period, the actual index i_t exceeds the trigger level (120% for Class B, 115% for Class C, 110% for Class D). When i_t exceeds the exhaustion level, the investors will lose the whole principal. Among three classes, Class D is exposed to the highest catastrophe mortality risk given the lowest attachment and detachment levels while Class B has the highest seniority due to the highest trigger (or attachment) and exhaustion (or detachment) levels.

There are four possible risk periods over the term of Vita II. However, a year can only count once toward a loss event. Because of the restriction on a year counting toward more than one loss event, there are only two possible events over the period. For example, if 2006-2007 resulted in a loss then the only other loss would be 2008-2009. No loss would be possible in 2007-2008 since 2007 was already considered. Furthermore, the coupon is based on the outstanding principal, not the original principal, so it reflects principal reductions. The details of the issuance are summarized in Table 4.

Table 4

Summary of the 2005 Swiss Re Vita II Mortality Bond

	Class B	Class C	Class D
Class size	$62 million	$200 million	$100 million
Trigger level	120%	115%	110%
Exhaustion level	125%	120%	115%
Coupon (bps)	LIBOR + 90	LIBOR + 140	LIBOR + 190

The percentage loss of the principal in year *t*, $loss_t$, is determined as follows,

$$loss_t = \min\left\{\frac{i_t - a}{d - a},\ 100\%\right\},$$

where a is the trigger level and d is the exhaustion level. Therefore, the payment at maturity of each class equals

$$\text{Payment at maturity}_i = \text{Class size}_i \times \begin{cases} 100\% - \sum_{t=2006}^{2010} loss_t & \text{if } \sum_{t=2006}^{2010} loss_t < 100\% \\ 0 & \text{if } \sum_{t=2006}^{2010} loss_t \geq 100\% \end{cases},$$

where *i*= B, C, or D.

Following Lin and Cox (2008), we apply the Wang transform to price mortality securities and create the transformed loss distribution based on the US population mortality index:[7]

[7] Our estimation serves as the upper bound on the market price of risk λ for the 2005 Swiss Re Vita II mortality bond. This arises from the fact that we use the US population index as the benchmark in this example while the

$$F_{\lambda}^{*}(L_i) = G[\Phi^{-1}(F(L_i)) - \lambda], \tag{3.4}$$

where L_i is the percentage loss of coupon and principal of class *i*. To estimate the market price of risk λ, we first simulate q_t with 10,000 trials based on the RS-GBM model and Lin and Cox (2008)'s model (3.3) respectively and then we calculate L_i. Based on equation (3.4), the coupon rates of different classes, the US population mortality index from 1901 to 2005 and the US Treasury rate term structure on December 28, 2005, our estimated market prices of risk of the Swiss Re Vita II are shown in Table 5.

Table 5

Market Price of Risk Based on the RS-GBM Model and Lin and Cox (2008)'s Model

Market Price of Risk λ	Class B	Class C	Class D
RS-GBM Model	0.8504	0.7796	0.5564
Lin and Cox (2008)'s Model	1.5343	1.3996	1.0884

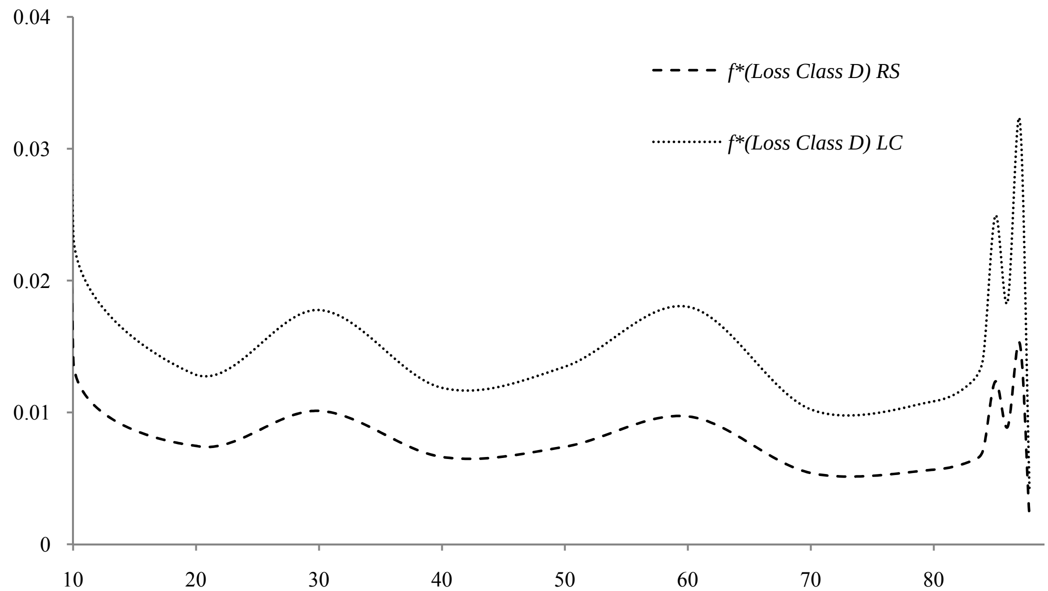
In both models, the tranche with a higher trigger level has a higher λ than that with a lower one. This is consistent with the findings of Froot and O'Connell (2008). The higher layer of mortality catastrophe risk is more difficult to predict. Froot and O'Connell (2008) state that higher uncertainty and opacity require a greater return (or higher λ) for bearing such risk.

Furthermore, Lin and Cox (2008)'s model has higher λ's than the RS-GBM model for all tranches. This can be explained by the shortcoming of the Lin and Cox (2008) model: underestimating the magnitude and the volatility of mortality risk. In the Lin and Cox (2008) model, each jump event is a one-time event that only lasts for one year. It means the mortality curve after the jump will go back to the original level and "is independent of that [mortality rate] during the shock" (Lin and Cox 2008). Although their method proposes a novel way to model unanticipated temporary death shock, it cannot capture mortality jumps that span more than a year. Accordingly, it underestimates the magnitude of those events and causes a lower expected non-transformed loss. Given the par spread of each tranche, it implies a notably higher market

Swiss Re deal is based on the weighted average of five developed countries. If we use the weighted index, we expect that our calculated λ will be lower than the one in this example because of the diversification effect of mortality risks among these five countries.

price of risk λ. That is, its transformed probability of L_i when $L_i > 0$ is higher. Take, as an example, Figure 3 shows the transformed loss distribution of class D when $L_D > 0$.

Figure 3
Transformed Loss Distributions of Class D



The horizontal axis in Figure 3 represents the loss of coupons and principal of Class D in millions of dollars and the vertical axis is the probability. The broken line (called "*f*(Loss Class D) RS*") denotes the transformed probability density function (PDF) of the loss with $\lambda = 0.5564$ by using the RS-GBM model and the two-factor Wang transform (3.4). Figure 3 illustrates that the Lin and Cox (2008) model, given the observed par spread of 190 bps of Class D, overestimates the probability of having a catastrophe death event since its transformed probability distribution shown as the dotted line called "*f*(Loss Class D) LC*" is above that based on the RS-GBM model. It is equivalent to say that the calculated risk premium of Class D will be lower for the Lin and Cox (2008) model because it underestimates loss magnitude and volatility.

As a robustness check, we price the 2003 Swiss Re Vita I mortality bond, the first mortality bond, using both methods.[8] Vita I is a three-year bond issued in 2003 by the Swiss Re Company. If the mortality index from 2004 to 2006 exceeds 130% of the actual 2002 level, then the investors will have a reduced principal payment. We reach the same pattern as that of the 2005

[8] For the details of 2003 Swiss Re Vita I mortality bond structure, see Lin and Cox (2008).

Swiss Re Vita II. Given the par spread 135 bps of Vita I, the market price of risk, again, is higher with the Lin and Cox (2008) model (λ=1.9220) than the RS-GBM model (λ=1.5525) based on the two-factor Wang transform, the US population mortality index from 1901 to 2002 and the US Treasury yield curve on December 30, 2003. Failure to model different regimes leads to a notable deviation from the right market price of risk and the correct transformed distribution. We conclude that modeling changes in mortality regimes plays an important role in the mortality securitization modeling.

It is interesting to point out that although three tranches of Vita II all have lower trigger levels (120% for Class B, 115% for Class C, 110% for Class D) than that of Vita I single tranche (130%), the par spread of Vita II (135 bps) exceeds that of Vita II Class B (90 bps). According to Morgan Stanley, the first mortality bond, Vita I, overcompensated the investors for their taking catastrophe mortality risk, so "the appetite for this security [the 2003 Swiss Re Vita I bond] from investors was strong". After the market got acquainted with this new type of security, the risk premia go down as shown in the reduced par spreads of Vita II. Furthermore, our pricing results, either based on the RS-GBM model or Lin and Cox (2008)'s model, confirm that the market price of risk λ of each tranche of Vita II is lower than that of Vita I.

3.6 Different Opinions on Future Mortality Evolution

Given the assumption that future mortality rates will follow trends captured in the period 1901 to 2005, the rates can be simulated from the distribution based on the RS-GBM model in Section 3.5. However, future mortality rates may deviate from what the US population mortality data from 1901 to 2005 suggest. Researchers may have conflicting opinions about future mortality evolution, in which case they can apply our RS-GBM model in Section 3.2 on a dataset that better reflects their expectations of future mortality (i.e. add/remove extreme events, or extend periods of high/low volatility). For instance, population mortality rates fluctuate more significantly before 1950 because of infectious diseases such as respiratory diseases. Similar scares have surfaced in the news over the past few years.[9] To illustrate future possible mortality evolution, in Table 6 we provide parameter values of the RS-GBM and GBM models estimated over the period 1901-1950, 1901-1960, 1901-1970 and 1901-1980 respectively.

[9] Examples are the SARS outbreak in 2003 and H1N1 flu in 2009.

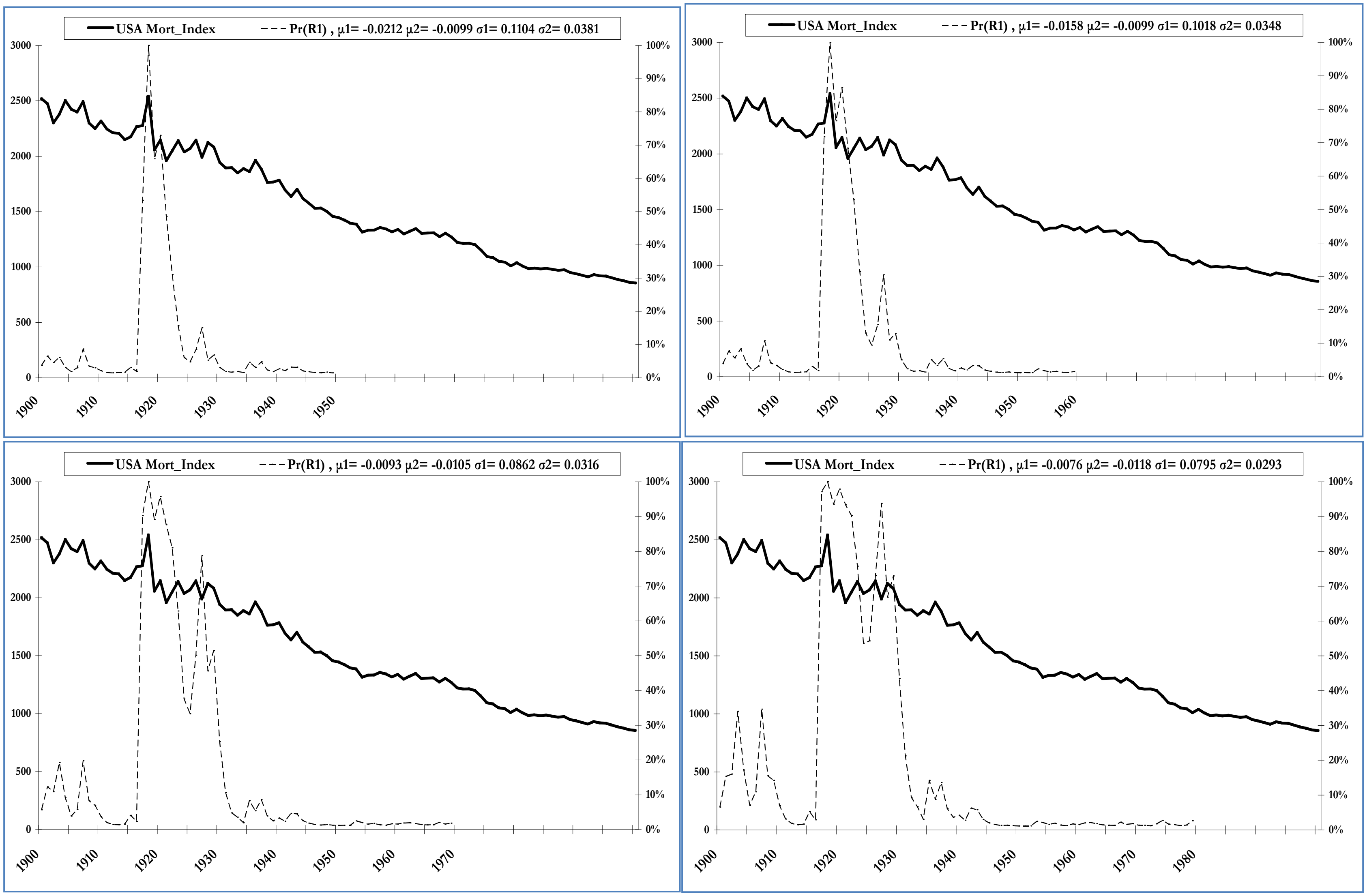
Table 6

Parameter Estimation and Out-of-Sample Forecasting of the RS-GBM and GBM Models Based on the Annual US Population Mortality Index from 1901 to 1950, 1960, 1970 and 1980 Respectively.

Statistics	1901-1950 (1)	1901-1960 (2)	1901-1970 (3)	1901-1980 (4)
RSGBM				
μ_1	-0.0212	-0.0158	-0.0093	-0.0076
μ_2	-0.0099	-0.0099	-0.0105	-0.0118
σ_1	0.1104	0.1018	0.0862	0.0795
σ_2	0.0381	0.0348	0.0316	0.0293
p_{12}	0.2818	0.2181	0.1336	0.1051
p_{21}	0.0309	0.0251	0.0216	0.0196
Log Likelihood	83.04	105.18	127.43	139.16
GBM				
α	-0.0111	-0.0105	-0.0103	-0.0111
σ	0.0510	0.0473	0.0446	0.0425
Log Likelihood	77.88	97.98	118.34	150.28
Better Model Based on				
Likelihood Ratio Test	RSGBM	RSGBM	RSGBM	RSGBM
Out of Sample Forecasting *Mean Absolute Error (Average Percentage Deviation from 2005 Actual Observation)*				
RSGBM	203.10 (24.49%)	170.02 (20.50%)	139.89 (16.87%)	97.21 (11.72%)
GBM	214.28 (25.84%)	181.73 (21.91%)	156.46 (18.87%)	105.24 (12.69%)

Figure 4

Estimation of the RSGBM Model on the Annual US Population Mortality Index for Periods 1901-1950, 1901-1960, 1901-1970 & 1901-1980.



The RSGBM model provides better fitness when estimating the model from 1901 to 1950 and its goodness of fit further improves as more observations with structurally different characteristics are included in the sample (Table 6). The volatility $\sigma_1 = 0.1104$ (or $\sigma_2 = 0.0381$) in column (1) of Table 6 is higher than that in Table 2 and in columns (2), (3), (4) of Table 6 since more extreme changes in mortality rates are reported in the first fifty years. Specifically, the period around 1918 is captured in a high volatility regime whose duration depends on the sample period. As more observations of low volatility enter the data, the duration of the high-volatility regime around 1918 and the relative difference in volatility parameters of the two regimes varies (see Table 6 and Figure 4).

We also conduct different out-of-sample forecasting for the RSGBM model based on each of the four sets of parameter estimates in Table 6.[10] For each of the four sample periods, we produce forecasts for the number of deaths per 100,000 U.S. population each year from year $t+1$ to 2005 for both of the RSGBM and the GBM models where t is the last year of the estimation sample period (i.e. t = 1950, 1960, 1970, or 1980). We simulate 10,000 paths of the mortality index and measure the mean absolute error and the average percentage deviation from the 2005 actual mortality index (in parentheses) across all years and paths. The results are reported at the bottom section of Table 6. The RSGBM model consistently has a lower mean absolute error and average percentage deviation in all specifications, which suggests that it has a better forecasting power than the GBM model.

4. Improving Lee-Carter Model with RS Model

The Lee-Carter (1992) model assumes normality in the error term of the mortality common risk factor affecting all age cohorts across years. In the following, we improve the Lee-Carter (1992) model with the RS technique to allow for non-normality features such as multi-modality, skewness and excess kurtosis.

4.1 Lee-Carter (1992) model

[10] We would like to thank the referee for this suggestion.

Lee and Carter (1992) develop a model which captures the evolution of mortality in mutually exclusive age cohorts, while at the same time a time-series common risk factor, k_t, links all cohorts together. Let $q_{x,t}$ denote the one-year death rate for age x at time t which is modeled as the following exponential relationship:[11]

$$q_{x,t} = \mathrm{Exp}\left[a_x + b_x k_t + \varepsilon_{x,t}\right] \tag{4.1}$$

or in logarithmic form:

$$\ln\left(q_{x,t}\right) = a_x + b_x k_t + \varepsilon_{x,t}\,. \tag{4.2}$$

Parameters a_x and b_x are age-specific while k_t is time varying and $\varepsilon_{x,t}$ is a zero mean disturbance term. Model (4.2) is not unique hence two constraints are imposed to ease the estimation: $\sum_x b_x = 1$ and $\sum_t k_t = 0$. The second constraint implies that a_x simplifies to the empirical average of age x over time:

$$a_x = \frac{\sum_t \ln\left(q_{x,t}\right)}{t} = \ln\left(\hat{q}_x\right). \tag{4.3}$$

Therefore, equation (4.2) can be re-written in terms of the mean log-mortality rate:

$$\ln\left(\tilde{q}_{x,t}\right) = \ln\left(q_{x,t}\right) - \ln\left(\hat{q}_x\right) \tag{4.4}$$

with a zero-mean error term $\varepsilon_{x,t}$, i.e.:

$$\ln\left(\tilde{q}_{x,t}\right) \sim N\left(\bar{\mu}_{x,t}, \sigma^2\right), \text{ and } \varepsilon_{x,t} \sim N\left(0, \sigma^2\right) \tag{4.5}$$

which can be simplified to:

$$\mathrm{E}\left[\ln\left(\tilde{q}_{x,t}\right)\right] = \bar{\mu}_{x,t} = b_x k_t\,. \tag{4.6}$$

One-year death rates in equation (4.4) are shown below in matrix form where time t, $(t = 0,1,2,...,T)$, is represented in columns while age x, $(x = 1,2,...,\omega)$, is represented in rows:

[11] Lee and Carter (1992) model the central death rate $m_{x,t}$. In most cases, the one-year death rate $q_{x,t}$ is very close to $m_{x,t}$. The main reason why we use $q_{x,t}$ instead of $m_{x,t}$ is that later when we price longevity securities, we use $q_{x,t}$ to calculate the security payoffs. If we used $m_{x,t}$ to estimate the Lee-Carter model, we had to make the distribution assumption to convert $m_{x,t}$ to $q_{x,t}$ for pricing. To minimize the assumptions, we use $q_{x,t}$ throughout the paper.

$$\ln(\tilde{q}) = \begin{bmatrix} \ln(\tilde{q}_{1,0}) & \ln(\tilde{q}_{1,1}) & \ln(\tilde{q}_{1,2}) & \ldots & \ln(\tilde{q}_{1,T}) \\ \ln(\tilde{q}_{2,0}) & \ln(\tilde{q}_{2,1}) & \ln(\tilde{q}_{2,2}) & \ldots & \ln(\tilde{q}_{2,T}) \\ \ln(\tilde{q}_{3,0}) & \ln(\tilde{q}_{3,1}) & \ln(\tilde{q}_{3,2}) & \ldots & \ln(\tilde{q}_{3,T}) \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ \ln(\tilde{q}_{\omega,0}) & \ln(\tilde{q}_{\omega,1}) & \ln(\tilde{q}_{\omega,2}) & \ldots & \ln(\tilde{q}_{\omega,T}) \end{bmatrix} = \begin{bmatrix} b_1k_0 & b_1k_1 & b_1k_2 & \ldots & b_1k_T \\ b_2k_0 & b_2k_1 & b_2k_2 & \ldots & b_2k_T \\ b_3k_0 & b_3k_1 & b_3k_2 & \ldots & b_3k_T \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ b_\omega k_0 & b_\omega k_1 & b_\omega k_2 & \ldots & b_\omega k_T \end{bmatrix}$$

The estimation is completed in two steps according to Lee and Carter (1992). In the first step singular value decomposition of the matrix is used to obtain estimates for $b_x, (x = 1,2,...,\omega)$, and $k_t, (t = 1,2,...,T)$. In the second step the time-series evolution of k_t is recalculated based on the actual number of deaths in year *t*. For mortality projection, k_t is assumed to follow a random walk with drift

$$k_t = k_{t-1} + g + e_t, \tag{4.7}$$

where g is a constant and e_t is a normally distributed error term with zero mean. Lee and Carter (1992) also include a dummy variable in their estimation to capture the one-time flu event of 1918 (not shown in equation (4.7)).

In our model, instead, we investigate the evolution of the error term in equation (4.7) as a RS process. We show that e_t escapes normality boundaries and therefore biases future forecasts of k_t. RS models allow a great deal of flexibility in defining the distribution attached to each state, a feature that we will explore next. Given the attractive features of RS models we will show how the RS model improves the fitness and forecasting power of k_t.

4.2 Modeling the Time-Series Common Risk Factor of the Lee-Carter (1992) Model as a RS Process

We first estimate the Lee and Carter (1992) model to obtain the a_x, b_x and k_t. Second, we model the time series evolution of mortality common factor, k_t, following equation (4.7). After running the model, we obtain

$$k_t = k_{t-1} - \underset{(0.064)}{0.2032} + e_t, \tag{4.8}$$

with the standard error shown in parenthesis. Model (4.8) means that the mortality common factor k_t, on average, decreases by g = -0.2032 per year. Lee (2000) ignores the error term when forecasting k_t. However, we believe that it might be more appropriate to have a closer look at its distribution.

Table 7
Descriptive Statistics of Residual Error Term from

Descriptive Statistics	Error e_t
Mean	0.0000
Standard Error	0.0640
Median	-0.0377
Standard Deviation	0.6522
Sample Variance	0.4254
Kurtosis	8.5164
Skewness	-1.3093
Range	5.7403
Minimum	-3.5551
Maximum	2.1852
Number of observations	104

This is evident from Table 7 where the summary statistics for the error term e_t from the model (4.8) are reported. The kurtosis of e_t is 8.516 while its skewness is negative, implying the error e_t may be deviating from normality. Therefore our next step is to test for non-normality deviations in the time-series common risk factor using RS model (2.2).

The first step in fitting a RS model is to rearrange equation (4.7) such that the first difference, Δk_t, of the time-series common risk factor becomes:

$$\Delta k_t = k_t - k_{t-1} = g + e_t \tag{4.9}$$

which is subsequently assumed to switch between two regimes after de-trending the series by the constant g:[12]

$$\Delta k_t - g = \begin{cases} {}^1e_t & if \;\; \rho_t = 1 \\ {}^2e_t & if \;\; \rho_t = 2 \end{cases}, \tag{4.10}$$

where

$${}^1e_t = \mu_1 t + \sigma_1 W_t \;\text{ and }\; {}^2e_t = \mu_2 t + \sigma_2 W_t \,. \tag{4.11}$$

Our model of choice shows that the means of the two regimes can be set equal to each other $(\mu_1 = \mu_2)$ as the improvement in goodness of fit tests from the equal-mean model to the unequal-mean model does not justify the use of separate means. In Panel A of Table 8 we present the maximum likelihood estimates of the two models: the normal error term as assumed by Lee and Carter (1992) and the RS-normal error term as proposed by our model. The goodness of fit tests in Panel B show overwhelming support for the RS-normal error model where its overall error mean is positive and the volatility in the high-volatility regime is much higher (1.8873) than that in the normal error model (0.6491) and in the low-volatility regime (0.4889).

Table 8

Maximum Likelihood Parameter Estimates of Competing Models

Panel A - Parameter Estimates

Estimates	Normal e_t	RS-normal e_t
μ_1	0.0000	0.0126
μ_2		0.0126
σ_1	0.6491	0.4889
σ_2		1.8873
p_{12}		0.0156
p_{21}		0.2790
Log-likelihood	-102.62	-86.32

[12] The estimation of parameter g and the parameters of the Regime Switching model is done simultaneously. The estimated value of g in equation (4.10) is the same as that estimated from equations (4.8).

Panel B - Goodness of Fit Tests for Competing Error Models

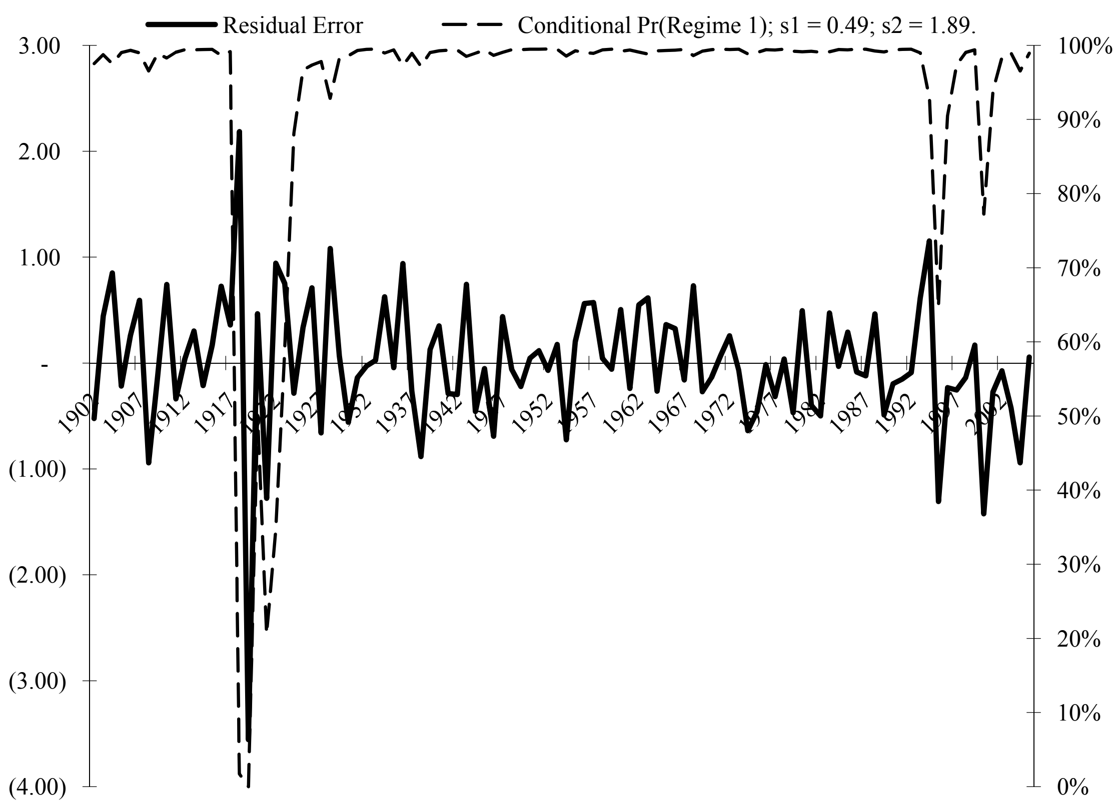
	Akaike IC	Schwartz Bayes	LL[*]
Normal e_t	-104.62	-107.27	-102.62
RS-Normal e_t	-92.32	-100.26	-86.32
LR Test[**]			32.60
p-value			<0.001
Better Model	RS-normal	RS-normal	RS-normal

[*] Log-likelihood value; [**]Likelihood-ratio test.

Figure 5 shows the time-series propagation of the error and the annual conditional probability of the error term being categorized in regime 1 (low volatility regime). It is evident from the chart that the influenza period introduced disturbances in the error term that have been isolated by the RS-normal model in a new short lived state of higher volatility. The short duration of this regime is evident from the probability transition matrix which implies a long-term probability of the process being in regime 2 is only 5.29% $\left(\pi_2 = p_{12}/\left(p_{12} + p_{21}\right)\right)$.

Figure 5

Conditional Probability of Error Term e_t Classified in Low Volatility Regime



In addition, it is important to note that a flu dummy may not be sufficient to capture the extreme impact of the respective event since there is a lasting effect beyond the time period that the flu effect was introduced in 1918. We observe that even though the event was short-lived, the resulting disturbance in volatility lasted for about 5 years. It implies that, in addition to the jump probability, we should also consider the evolution of mortality volatilities in an extreme event.

5. Pricing Longevity Securities with RS Models

Insurers and pension plans manage mortality/longevity risk for various reasons. For example, mortality/longevity risk management reduces the expected cost of financial distress, reduces the expected tax payments when the insurers face a convex tax function, and reduces the probability of costly external financing if the catastrophe mortality/longevity events exhaust the insurers' internal funds. The traditional way for the insurers to hedge mortality risk is to purchase reinsurance. Compared with the reinsurance markets, the capital markets have a much bigger capacity to pay potential catastrophe losses (Lin and Cox 2005). As such, insurance–linked securitization, as an alternative risk management method, has gained more and more attention from both scholars and practitioners. Since 2003, the life insurers have successfully transferred catastrophe mortality risk to the capital market by issuing several death-linked securities. The market appetite for those death-linked securities is strong (Lin and Cox, 2008). The success of death-linked securities, at least, is attributed to attractive security designs. But the market for longevity securities has not developed. A market for longevity securities will develop if their designs and prices are attractive to potential buyers and sellers. In this section, we first introduce a feasible design for longevity options that aim at covering unexpected dramatic mortality improvement risk based on a population index, and then show how our mortality RS model can be used in this longevity security pricing. In Section 3, we have shown how to incorporate the RS model to the Wang transform to price the Swiss Re Vita I and Vita II mortality bonds. The RS model can nicely fit other pricing approaches. To illustrate this, in the following discussion, we price longevity options with another pricing method, the Esscher transform, based on the mortality dynamics forecasted by the RS model.

5.1 Longevity Options

Most of the payoffs of the existing death-linked securities are based on the indices calculated from publicly available mortality data. For instance, the principals and/or coupons of three series of Swiss Re mortality bonds (Vita I in 2003, II in 2005 and III in 2006) depend on the population mortality experiences of five countries.[13] Class B ($80 million) of the Tartan Capital Ltd. death-linked notes is based on the US population mortality index, which is defined as the weighted average of death rates of different population age groups reported by the Centers for Disease Control and Prevention (CDC). The advantage of using population mortality indices is that they are transparent to investors and hedgers, thus reducing moral hazard problems and increasing liquidity.

Using population mortality indices as the underlying for death-linked securities seems well-received in the market. We expect they will also work for potential longevity securities. As such, the longevity options discussed in this paper are based on the population longevity indices proposed by Cox et al. (2008), which can be easily calculated from the publicly available databases, such as the Human Mortality Database.

Longevity risk has a less dramatic process but its effect lasts a much longer period than that of mortality risk. For annuity insurers or pension plans, they are more concerned about the accumulation effect of longevity risk in the long run since they can predict the number of survivors with high accuracy in the short term, say in the next five or ten years. Furthermore, the annuity insurers and pension plans may be interested in hedging longevity risk associated with given cohorts. To illustrate, consider the following example.

At time 0, an annuity insurer, called XYZ Insurance Company, has a cohort of N annuitants all aged x = 65. XYZ promises to pay a survival benefit B per annuitant as long as the annuitant is alive at the end of year t (t =1, 2, 3, ...). Accordingly, given the information $\Re_0$ up to time 0, XYZ expects to pay the survival benefit P_t

$$\mathrm{E}[\mathrm{P}_t] = N \times B \times \mathrm{E}[{}_t\widetilde{p}_{65} \mid \Re_0]$$

[13] Those population mortality data are available on the website, www.mortality.org.

in year *t*. The *t* -year survival rate

$$ {}_t\tilde{p}_{65} = \tilde{p}_{65,1} \times \tilde{p}_{66,2} \times \cdots \times \tilde{p}_{65+t-2,t-1} \times \tilde{p}_{65+t-1,t}, $$

where $\tilde{p}_{65+t-1,t}$ represents the annuitant's one-year survival rate in year *t* given he/she survives $t-1$ years.

XYZ is concerned that the realized ${}_t\tilde{p}_{65}$ in year *t* is much higher than its expectation $\mathrm{E}[{}_t\tilde{p}_{65} \mid \Re_0]$ at *t* = 0, so it is interested in managing longevity risk of this 65-year old cohort. If XYZ is worried about, for example, the risk that the realized ${}_t\tilde{p}_{65}$ is higher than two standard deviations above $\mathrm{E}[{}_t\tilde{p}_{65} \mid \Re_0]$, it can purchase a longevity option similar to that in Cox et al. (2008).

In Cox et al. (2008), the underlying of the *t*-year European longevity call option is *t*-year survivor rate for age *x*, ${}_t\tilde{p}_x$. If the realized ${}_t\tilde{p}_x$, denoted ${}_t p_x$, in year *t* is higher than the strike level ${}_t\bar{p}_x$, the call owner will get a positive payoff. Otherwise the value of the call is zero. More specifically, given a notional amount NA, the payoffs to the call owner in year *t* are:

$$ V_{x,t} = \mathrm{NA} \times \begin{cases} {}_t p_x - {}_t\bar{p}_x & if \quad {}_t p_x > {}_t\bar{p}_x \\ 0 & if \quad {}_t p_x \le {}_t\bar{p}_x \end{cases}. \tag{5.1} $$

The potential buyers of the longevity call option could be annuity insurers and pension plans. When the covered cohorts experience an unexpected mortality improvement, the annuity insurers or pension plans will be forced to pay much higher survival benefits. If they purchase longevity call options, the longevity event will trigger the payments of the calls. The positive payoffs from the options will offset the excess payments to the annuitants, thus hedging longevity risk. To reduce moral hazard and increase liquidity, the underlying mortality index should be based on relatively frequent mortality studies or assessments of a reference population, such as the Human Mortality Database. From those widely accepted sources, we can obtain all realized values of

$$ \tilde{p}_{x,1}, \tilde{p}_{x+1,2}, \cdots, \tilde{p}_{x+t-2,t-1}, \tilde{p}_{x+t-1,t} $$

as time passes by. At the end of the t^{th} year, ${}_t p_x$ will be calculated from the following expression

$$ {}_t p_x = p_{x,1} \times p_{x+1,2} \times \cdots \times p_{x+t-2,t-1} \times p_{x+t-1,t}, $$

where $p_{x+t-1,t}$ is the realized one-year survival rate for age $x+t-1$ based on the t^{th} year mortality table.[14]

We can set a reasonable strike level based on our current expectation, $\mathrm{E}[{}_t\tilde{p}_x \mid \Re_0]$. $\mathrm{E}[{}_t\tilde{p}_x \mid \Re_0]$ can be estimated from our improved Lee-Carter model with the RS error term, introduced in Section 4. Then the strike level ${}_t\bar{p}_x$ could be defined as *n* standard deviations above the mean,

$$ {}_t\bar{p}_x = \mathrm{E}[{}_t\tilde{p}_x \mid \Re_0] + n \times \sigma_{{}_t\tilde{p}_x \mid \Re_0}, $$

where $\sigma_{{}_t\tilde{p}_x \mid \Re_0}$ is the standard deviation of ${}_t\tilde{p}_x \mid \Re_0$.

Related to the above example, if XYZ feels comfortable to take longevity risk for the first ten years and only worries about the uncertainties after that, it can purchase various *t*-year longevity European call options based on different ${}_t\tilde{p}_{65}$ where $t > 10$, to hedge the long-term longevity risk associated with its aged-65 cohort. XYZ can choose the strike level ${}_t\bar{p}_{65}$ based on its risk attitude and financial status. Firms usually set a higher strike level when they are less risk averse and well capitalized.

In the above example, we assume that XYZ currently has only one cohort at the age of 65, so it purchases the longevity option on age 65. This example can be generalized to the situation where insurers have multiple-age groups. To hedge longevity risk, the insurers can purchase a combination of various longevity call options on different ages. Moreover, the above discussion is also applied to pension plans. In summary, the longevity options are flexible enough to fit the needs of different annuity insurers and pension plans by varying the age *x*, the term *t*, and the strike level ${}_t\bar{p}_x$.

[14] Because of lag in the availability of mortality data, the settlement year and the reference year may differ. To simplify our example, without loss of generality, we assume the settlement year and the reference year are the same.

5.2 Pricing Longevity Securities

In addition to designing attractive longevity securities, pricing those products correctly is also crucial to the success of longevity securitization. In Sections 3 and 4, we have shown the attractive features of RS models in modeling mortality stochastic process. RS models are also flexible enough to fit different pricing frameworks. Here, we show how to price a longevity option with the RS model and a widely accepted pricing method in the insurance and actuarial science literature, the Esscher transform.

5.2.1 Forecasting Mortality Rates with RS Models

In producing forecasts for the *T*-year ahead mortality rates $q_{x,t}$ (t = 1, 2, ... , T), we first simulate 10,000 observations of e_t from the RS-normal model (4.10) each year for *T* years. Then the common risk factor k_t in year t (t = 1, 2, ... , T) is predicted by adding the constant $g = -0.2032$ to each simulated RS-normal error term e_t. To compare our results to those based on the normal assumption, we also forecast k_t based on the 10,000 simulated normal error terms for year *t*.

Central to the pricing of mortality and longevity securities is the simulation of survival and death probabilities. We assume that the one-year death probability for age *x* in year *t* can be approximated by $q_{x,t}$ constructed from equation (4.1) with the estimated values of a_x, b_x and the time-series forecast of k_t. For example, the one-year ahead one-year death rate for a 66-year old (x = 66 and t = 1) would be equal to:

$$q_{66,1} = \text{Exp}\left[a_{66} + b_{66} * k_1\right]$$

where $$k_1 = k_0 - 0.2032 + e_1$$

and e_1 is a random draw from

$$\text{Normal}\left(0.0000,\ 0.6491\right),$$

if e_t is assumed normal, or e_1 is a random draw from

RS - Normal $\left(\mu_1 = 0.0126, \mu_1 = 0.0126, \sigma_1 = 0.4889, \sigma_2 = 1.8873, p_{12} = 0.0156, p_{21} = 0.2790\right)$,

if the error term is assumed to be switching between two normal distributions, with the same mean and different deviation.

5.2.2 Esscher Transform

The Esscher transform provides an established method to derive security prices when the market is incomplete. In incomplete market models, there are many equivalent martingale measures. Among various methods, the Esscher transform is accepted as a very useful technique to obtain a reasonable equivalent martingale measure.

Definition 1. *Let X denote a risky asset. Then the Esscher transformed probability measure of P associated with the process X and a constant c given the information set $\Re$ is the probability measure Q, which is defined as*

$$\frac{\mathrm{d}Q}{\mathrm{d}P} \mid \Re = \frac{e^{-cX}}{\mathrm{E}[e^{-cX}]}.$$

The constant c has economic meanings. If the equivalent martingale measure corresponds to the exponential utility function

$$u(y) = \frac{1}{c}\left(1 - e^{-cy}\right), \tag{5.2}$$

the parameter c is the absolute risk aversion coefficient. Suppose the current value of X is x_0. The parameter c must satisfy the following condition

$$x_0 = e^{-r_f} \frac{\mathrm{E}[Xe^{-cX}]}{\mathrm{E}[e^{-cX}]},$$

where r_f is the annual risk-free rate. Given the value of c, the price of a derivative security with payoff $V(X)$, in the sense of Bühlmann, equals

$$e^{-r_f} \mathrm{E}_Q\left[V(X)\right] = e^{-r_f} \frac{\mathrm{E}[V(X)e^{-cX}]}{\mathrm{E}[e^{-cX}]} = x_0 \frac{\mathrm{E}[V(X)e^{-cX}]}{\mathrm{E}[Xe^{-cX}]}. \tag{5.3}$$

For t-year longevity options on age x, the underlying X is the t-year survival rate ${}_t\tilde{p}_x$ and the payoff $V(X)$ is determined by the relation between ${}_t\tilde{p}_x$ and the strike level ${}_t\bar{p}_x$. The value of $V(X)$ at year t, $V_{x,t}$, is specified in equation (5.1).

We can use the RS-normal model (4.10) to model and simulate $X={}_t\tilde{p}_x$ and $V_{x,t}$, following the procedures described in Section 5.2.1. Then the risk aversion parameter c in the exponential function can be estimated with the indifference pricing methods using, for example, annuity market quotes. Suppose the risk aversion is the same in annuity markets and in longevity security markets. We can use the estimated c from annuity markets to price longevity securities.

According to the indifference pricing method, the appropriate single-premium immediate life annuity (SPIA) premium ξ for an insurer with the initial wealth w is the one that satisfies

$$\mathrm{E}\left[u\left(w+\xi-B\sum_{t=0}^{\infty}v^t{}_t\tilde{p}_x\right)\right]=u(w), \tag{5.4}$$

where v^t is the t-year discount factor and B is the annual survival benefit. Equation (5.4) suggests that ξ should be set in such a level that the insurer will be indifferent between not selling this contract (in this case, the wealth w is certain) and selling the contract (in this case, the wealth $w+\xi-B\sum_{t=0}^{\infty}v^t{}_t\tilde{p}_x$ is random). This is called the *principle of equivalent utility* (See Gerber and Pafumi (1998)). If the utility function of this insurer is an exponential function (5.2), we can obtain an explicit expression for the premium ξ as follows,

$$\xi=\frac{1}{c}\ln\mathrm{E}\left[e^{cB\sum_{t=0}^{\infty}v^t{}_t\tilde{p}_x}\right].$$

Immediate annuity sales in the U.S. were a total of ξ = \$5.3 billion in 2005 with an average monthly payout rate of \$6.50 per premium \$1,000 (Fenton and Taht 2007; Stern 2008). We assume the expense factor 12.25 percent and the annual interest rate of 4.5 percent. Assuming constant force of mortality, we can simulate ${}_{t/12}\tilde{p}_x$ with the RS model. Accordingly, the average risk aversion parameter c in the SPIA market is calibrated to the SPIA market price ξ from the following equation,

$$\xi = \frac{1}{c} \ln \mathrm{E}\left[e^{cB\sum_{t=0}^{\infty} v^{t/12}\, {}_{t/12}\tilde{p}_x} \right]$$
$$5{,}300{,}000{,}000 \times (1-.1225) = \frac{1}{c} \ln \mathrm{E}\left[e^{c \times 6.5 \times \frac{5{,}300{,}000{,}000}{1{,}000} \times \sum_{t=0}^{\infty} v^{t/12}\, {}_{t/12}\tilde{p}_x} \right]. \tag{5.5}$$

After simulating for 10,000 trials and solving equation (5.5), we obtain $c = 2\times10^{-8}$. Next, we price longevity call options using simulated survival rates ${}_t\tilde{p}_x$ and the Esscher transform (5.3) with the estimated risk aversion $c = 2\times10^{-8}$. We assume that the average risk aversion in the longevity security market is the same as that of the US annuity market. Tables 9 and 10 show the prices of the 15- and 20-year longevity call options on ${}_{15}\tilde{p}_{65}$ and ${}_{20}\tilde{p}_{65}$ respectively. The strike level equals two standard deviations or three standard deviations above the mean of the simulated ${}_t\tilde{p}_{65}$ where t = 15 or 20. To show the importance of modeling the different mortality states with the RS-normal model, we also report the prices based on the normal error distribution.

Table 9

15-year Longevity Call Option Premiums on ${}_{15}\tilde{p}_{65}$ per $100,000 Notional Amount

Strike Level	RS-Normal Premium	Normal Premium
0.6281	23.22	15.63
0.6458	6.70	0.56

Table 10

20-year Longevity Call Option Premiums on ${}_{20}\tilde{p}_{65}$ per $100,000 Notional Amount

Strike Level	RS-Normal Premium	Normal Premium
0.4650	28.97	21.49
0.4896	8.40	1.10

For example, given a notional amount (NA) of $100,000 and the strike level ${}_{20}\bar{p}_{65}$ equal to two-standard deviations above the mean of 10,000-trial simulated ${}_{20}\tilde{p}_{65}$ in the RS model,

$$ {}_{20}\overline{p}_{65} = \mathrm{E}[{}_{20}\widetilde{p}_{65} \mid \Re_0] + 2 \times \sigma_{{}_{20}\widetilde{p}_{65}|\Re_0} = 0.4155 + 2 \times 0.0246 = 0.4650, $$

the premium of the 20-year longevity call option $C_{65,20}$ on ${}_{20}\widetilde{p}_{65}$ is \$28.97 based on the RS model. This premium is only 0.17% of the expected survival payment in the 20th year

$$ 0.17\% = \frac{C_{65,20}}{\mathrm{NA} \times \mathrm{E}[{}_{20}\widetilde{p}_{65} \mid \Re_0] \times (1+r_f)^{-20}} = \frac{28.97}{100,000 \times 0.4155 \times (1+0.045)^{-20}} , $$

but it can cover the extreme longevity risk that ${}_{20}\widetilde{p}_{65}$ improves by more than two standard deviations above the current expectation. As the strike level increases, the call premium falls (e.g. \$8.40 at the strike level of 0.4896 vs. \$28.97 at the strike level of 0.4650 in Table 10) because of a lower probability of having a larger-scale longevity event that will trigger the call payment.

Furthermore, the pricing results in Tables 9 and 10 reflect the underestimation of catastrophic longevity risk in the model with the normal error assumption. The option premiums based on the normal error assumption are consistently and significantly lower than those based on the RS-normal model. For example, in Table 9, given the strike level 0.6281 the 15-year longevity call option premium on ${}_{15}\widetilde{p}_{65}$ is \$23.22 based on the RS-normal model but the option premium with the same strike level based on the normal error assumption is only \$15.63 (33% lower). This pattern is enforced as the strike level increases. Take Table 10 as the example. The option premium based on the RS-normal model (\$8.40) is 7.6 times the premium (\$1.10) based on the model with the normal error assumption. Under-pricing of the normal model arises from its failure to capture the fat tail.

Our longevity options in this paper are based on an index that is calculated from publicly available population mortality data. The same modeling and pricing techniques can be applied to other indices. The main advantage of using population mortality index is that it is more transparent thus reduces transaction costs and increases market liquidity. However, higher basis risk may arise as the hedger's business composition may differ from the population age composition. The basis risk from using population longevity index can be reduced by purchasing a combination of various longevity options with different time to maturity, cohorts and notional amounts. This will allow insurers and pension plans to replicate their risk exposure using a bundle of longevity options that would minimize their basis risk.

6. Conclusion

In this paper we construct RS models to investigate two dynamic mortality processes: the population mortality index and the time-series common mortality risk factor for different ages from the Lee-Carter (1992) model.

First, we looked at the US population mortality index to test for structural changes in the underlying death probability for all age cohorts from all death causes. We found that a RS model between two geometric Brownian motions (RS-GBM) not only provides a significant improvement over prior mortality models with jumps but has many attractive features. For instance, RS-GBM models provide a transparent representation of the timing of a structural change in death probabilities, the duration of each regime and finally paint a clear picture with respect to parameter estimates of the mean vs. volatility effect in aggregate mortality rates. We find that a high volatility regime of about three times the volatility of the low volatility regime (5.4% vs. 1.8%) is persistent for the first half of the century, while a lower volatility regime dominates after the Second World War. The overall mean-parameter of the two regimes points to the same direction as that of competing models. In addition to the statistical improvement provided by the RS-GBM model, we discover a lower implied market price of risk compared to that of other models when pricing the 2003 Swiss Re Vita I and the 2005 Swiss Re Vita II mortality bonds.

Our second application pertains to the time-series common risk factor, k_t, across all age cohorts in the popular model by Lee and Carter (1992). The time-series error term of k_t is once again modeled as a RS model, as we believe that disturbances introduced by extreme observations over time, force the error term to deviate from normality. We find that the error term is classified in a higher volatility normal regime for about 5% of the time. This takes place around periods of significant jumps in mortality rates (e.g. caused by the 1918 flu). During these periods the volatility is about quadruple the volatility value in the persistent lower volatility regime. We highlight the economic significance of our model by pricing longevity options with the Esscher transform based on the normal error and the RS-normal error models. We find that the option premiums based on the RS-normal error model are consistently and substantially higher than those based on the normal error model, a result that follows from the fatter tail in the RS-normal error distribution.

Appendix A

In this section we describe how the log-likelihood function of the RS model

$$Log[L(\Theta)] = \sum_{t=1}^{n} \log[f(y_t \mid \Theta, y_1, y_2, y_3, \ldots y_{t-1})]$$

can be estimated by first considering the contribution to the log-likelihood function of the t^{th} observation:

$$\log[f(y_t \mid y_{t-1}, y_{t-2}, y_{t-3}, \ldots y_1, \Theta)]. \tag{A.1}$$

This is calculated recursively for each year:

$$f(\rho_t, \rho_{t-1}, y_t \mid y_{t-1}, y_{t-2}, y_{t-3}, \ldots y_1, \Theta) =$$

$$p(\rho_{t-1} \mid y_{t-1}, \ldots y_1, \Theta) * p(\rho_t \mid \rho_{t-1}, \Theta) * f(y_t \mid \rho_t, \Theta) \tag{A.2}$$

where the transition probability between regimes is $p(\rho_t \mid \rho_{t-1}, \Theta)$. (A.3)

The probability density function of the distribution at each state of the Markov chain is

$$f(y_t \mid \rho_t, \Theta) = {}^{\rho_t} f(y) \tag{A.4}$$

And the probability function that characterizes transitions between the two regimes (Hamilton and Susmel, 1994) is:

$$p(\rho_{t-1} \mid y_{t-1}, \ldots y_1, \Theta) =$$

$$\frac{f(\rho_{t-1}, \rho_{t-2} = 1, y_{t-1} | y_{t-2}, y_{t-3}, \ldots y_1, \Theta) + f(\rho_{t-1}, \rho_{t-2} = 2, y_{t-1} | y_{t-2}, y_{t-3}, \ldots y_1, \Theta)}{f(y_{t-1} | y_{t-2}, y_{t-3}, \ldots y_1, \Theta)} \quad . \tag{A.5}$$

Finally we can calculate the value of $f(y_t | y_{t-1}, y_{t-2}, \ldots y_1, \Theta)$ as the sum over the 4 expected values under $\rho_t = 1,2$ and $\rho_{t-1} = 1,2$. An important property of discrete time Markov chains is that the probability of a process being in a specific state at time t depends only on where the process was at time $t-1$. To start the recursive process, we need to assign initial values to the parameter set Θ, and the probabilities of starting at regime 1 or 2, $p(\rho_o)$. These values can be set equal to the values of the invariant probability distribution π_1, π_2 :

$$[\pi_1, \pi_2]\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} = \begin{bmatrix} \pi_1 p_{11} + \pi_2 p_{21} = \pi_1 \\ \pi_1 p_{12} + \pi_2 p_{22} = \pi_2 \end{bmatrix} \tag{A.6}$$

which is also referred to as the unconditional probability distribution. Given that the probabilities of transitioning from one regime to another should add up to unity, it is easy to show $p_{11} + p_{21} = 1$ and $\pi_1 + \pi_2 = 1$. Consequently the entry in the first row and first column on the right hand side of equation (2.10) simplifies to:

$$\pi_1 = p_{21} / (p_{21} + p_{12}). \tag{A.7}$$

Therefore given Θ, we start the recursion at $t = 1$ with initial values

$$f(\rho_t = 1, y_1 | \Theta) = \pi_1 * {}^{\rho_t = 1}f(y_1) \tag{A.8}$$

$$f(\rho_t = 2, y_1 | \Theta) = \pi_2 * {}^{\rho_t = 2}f(y_1) \tag{A.9}$$

$$f(y_1 | \Theta) = f(\rho_1 = 1, y_1 | \Theta) + f(\rho_1 = 2, y_1 | \Theta), \tag{A.10}$$

which at $t = 2$ gives starting values for

$$p(\rho_1 | y_1, \Theta) = f(\rho_1, y_1 | \Theta) / f(y_1 | \Theta). \tag{A.11}$$

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