DISCUSSION PAPER PI-0815

Optimal Asset Allocation Strategy for Defined-Contribution Pension Plans with Power Utility

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June 2008

ISSN 1367-580X

The Pensions Institute
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City University
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http://www.pensions-institute.org/

Optimal asset allocation strategy for defined-contribution pension plans with power utility

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Abstract

Optimal asset allocation strategies of defined-contribution pension plans for members whose terminal utility is a power function of wealth-to-wage ratio is investigated in this paper. The portfolio problem is to maximize the expected terminal utility in the presence of three risk sources, interest risk, asset risk and wage risk. A closed form solution is found for the asset allocation problem and the optimal portfolio composition is horizon independent when there is no non-hedgeable wage risk or there is no further contribution from wage incomes. When future contributions from wage income are hedged by short-selling a wage replicating portfolio, the optimal composition of financial wealth on hand (i.e. pension portfolio wealth + short-sold wage replicating portfolio) is horizon-dependent. The optimal asset allocation strategy is equivalent to invest in two mutual funds, one of which is to hedge wage risk and the other a speculative fund to satisfy the risk appetite of the plan member.

Keywords: Defined-contribution pension plan; Wage risk; Optimal asset allocation; Power utility; Hamilton-Jacobi-Bellman equation

1. Introduction

The optimal asset allocation problem for defined-contribution (DC) pension plans can be viewed as a special form of consumption and portfolio problem. Most studies on the consumption and portfolio allocation strategies over multiple periods are built upon the classical dynamic optimization model by Merton (1969, 1971), which assumes constant interest rates and constant risk premiums without wage incomes. Since empirical studies show that stochastic variations in interest rates and in risk premiums exist, it may not be appropriate to assume a constant instantaneous interest rate in portfolios with a long horizon such as pension funds. Later studies extend Merton's work with stochastic interest rates (Sorensen, 1999; Liu, 2001; Campbell and Viceira, 2002) or stochastic risk premiums (Kim and Omberg, 1996). Studies on DC plan strategies generally assume stochastic interest rates.

With stochastic interest rates, the financial market is usually assumed to have three types of asset: cash, bonds and equities (stocks). Boulier et al (2001), Deelstra et al (2003) and Battocchio and Menoncin (2004) use these three assets in their studies on optimal asset allocation strategies for DC pension plans. In those studies, the stock price follows a geometric Brownian motion which includes volatilities from risk sources of both the interest rate and the stock market. Although stochastic interest rates make bonds distinct from cash and equities, in some studies bonds are not explicitly differentiated from other risky assets. Vigna and Haberman (2001) assume two assets: one low risk asset and one high risk asset. Haberman and Vigna later (2002) extended their assumption into a sequence of N assets with increasing returns and volatilities. Cairns et al (2006) use one risk-free asset and N risky assets, and the return on each risky asset follows a geometric Brownian motion with volatilities of N risk sources.

While Merton type consumption and portfolio studies often ignore wage incomes for computational simplicity (Merton 1969, 1971; Kim and Omberg 1996; Sorensen 1999; Liu 2001), in pension asset allocation studies wage incomes usually appear explicitly in the asset allocation problem. Contributions from wage income and returns from investing pension wealth are both important for the growth of pension wealth. Various treatments of wage income have been used in studies on asset allocation strategies for DC pension plan. Deterministic wage processes are used in Boulier et al (2001), Deelstra et al (2003), Vigna and Haberman (2001), and Haberman and Vigna (2002). Cairns et al (2006) and Battocchio and Menoncin (2004) used stochastic wage incomes, which are governed by geometric Brownian processes. The wage income process includes volatilities from risk sources of the interest rate and the stock market, with or without a non-hedgeable risk that is independent of financial market risk sources. Pension plan members will put a constant fraction of their wages into the pension fund.

One important difference between pension fund asset allocation problem and Merton's consumption and portfolio problem is that, the objective of pension plans is to maximize the terminal utility and there is no consumption or consumption-derived utility before retirement. The optimal allocation strategy for a DC pension plan depends critically on the specifications of terminal utility function. Power utility (Boulier et al 2001; Deelstra et al 2003; Cairns et al 2006), exponential utility (Battocchio and Menoncin 2004; Hendersen 2005) and quadratic disutility (Vigna and Haberman, 2001; Haberman and Vigna, 2002) have been used in solving the optimal asset allocation problem for DC pension plans. Since power utility is usually thought to be more consistent with empirical data, most DC pension plan studies have assumed a power function for terminal utility (Boulier et al, 2001; Cairns et al, 2006; Deelstra et al, 2003). With power utility, people make the same decision regardless of their wealth levels when the risks and the cost to avoid them are expressed as fractions of wealth.

Defined benefits (DB) pension plans have often been used as bench mark to assess the performance of DC pension plans. Pensions from defined benefits (DB) plans are generally based on final wages, implicitly assuming that pension income should be comparable to the existing standard of living. The intention to match DB plan benefits indicates implicitly the need to have pension income comparable to existing wages in DC plans and some role of habit formation (Spinnewyn 1981; Becker and Murphy 1988) in terminal utility. To relate terminal utility with the existing standard of living, Cairns et al (2006) assume that terminal utility is a function of wealth-to-wage ratio or replacement ratio (pension-to-final wage ratio). The use of replacement ratio is more appropriate for an individual who intends to convert her pension wealth into a life annuity on retirement, which suggests that she is more risk averse and perceiving life annuities as good value.

Using one risk-free asset and N risky assets, Cairns et al (2006) find that optimal asset allocation in risky assets needs three efficient mutual funds if the terminal utility is a function of replacement ratio. One mutual fund (which is heavily dominated with equities) is to satisfy the risk appetite of the plan member. The second fund (which is heavily dominated with cash) is to hedge the wage risk. The third fund (which is heavily dominated with bonds) is to hedge interest rate risk. Cairns et al (2006) call the three funds "equity", "cash" and "bond" fund respectively. If the terminal utility is a function of wealth-to-wage ratio, the optimal asset allocation needs only the "equity" fund and the "cash fund".

Although Cairns et al (2006) indicated that the "equity", "cash" and "bond" funds are heavily dominated by equities, cash and bonds respectively, they did not provide a measure on how to gauge the dominance. Is it possible that the "equity" fund is dominated by bonds in some scenarios? The present paper extends the study of Cairns et al (2006) by investigating the composition of those mutual funds. For simplicity, I assume that the pension plan can invest in three assets, cash, bond and stock (Boulier et al 2001; Deelstra et al 2003; Battocchio and Menoncin 2004) and that the terminal utility is a function of wealth-to-wage ratio. The assumption of wealth-to-wage ratio as the argument of terminal utility function is more appropriate for individuals who are reluctant to annuitize their pension wealth on retirement, which has been shown to be the case for most people (Brown and Warshawsky 2001).

In the present study I find that the optimal asset allocation in risky assets consists of three components: (i) a preference free component to hedge wage risk, (ii) a speculative component, proportional to both the portfolio Sharpe ratio and the inverse of the relative risk aversion index, and (iii) a hedging component depending on the state variable parameters. With the same assumptions on the wage process and risky assets as those in Battocchio and Menoncin (2004) and Cairns et al (2006), the third component (which corresponds to the "bond" fund of Cairns et al) disappears. The other two components all contain both bonds and equities, and the preference free hedging component (corresponding to the "cash" fund of Cairns et al) does not contain cash assets (because this is an investment in risky assets) if the proportion of investment in the riskless asset ($p_{A0} = 1 - \sum_{i=1}^{N} p_{Ai}$ in Cairns et al 2006) is not included.

This paper is organized as follows. Section 2 formulates the financial market, wage and pension wealth growth models. Section 3 presents the optimization problem and the Hamilton-Jacobi-Bellman equation. Section 4 solves the optimal asset allocation problem for power utility. Since optimization for power utility with non-hedgeable wage income risk cannot be solved in closed-form, I only solve the cases where pension contribution has stopped or wage risk is fully hedgeable. Section 5 discusses and summarizes the results in this paper.

2. The model

In this section I introduce the financial market structure, wage process and pension wealth process using wage as a numeraire.

2.1. Market structure

The specifications of the financial market are similar to those in Boulier et al (2001), Deelstra et al (2003) and Battocchio and Menoncin (2004). The financial market is frictionless and continuously open, with no arbitrage. There are three types of asset in the financial market: cash, bonds and equities. For simplicity, I assume only one equity asset, a stock, available, which can be considered as the index of a stock market. The uncertainty in the financial market is described by two standard and independent Brownian motions $Z_r(t)$ and $Z_S(t)$ with $t \in [0, T]$, defined on a complete probability space (W, F, P) where P is the real world probability. The filtration $F = F(t)$

$\forall t \in [0,T]$ generated by the Brownian motions can be interpreted as the information set available to the investor at time t.

The instantaneous risk-free rate of interest $r(t)$ follows an Ornstein-Uhlenbeck process (Vasicek model)

$$dr(t) = a(b - r(t))dt + s_r dZ_r(t),$$

$$r(0) = r_0. \qquad (1)$$

In equation (1), a and b are strictly positive constants, and s_r is the volatility of interest rate. The instantaneous drift $a(b - r(t))$ has an effect to pull the process towards its long term mean b with magnitude proportional to the deviation of the process from the mean (mean-reverting). The stochastic element $Z_r(t)$ causes the process to fluctuate in an erratic, but continuous fashion (Vasicek, 1977).

When the interest rate process is described by equation (1), the price of zero-coupon bonds for any date of maturity t at time t, $B(t, t, r)$, is governed by the diffusion equation (Vasicek 1977; Boulier et al 2001; Deelstra 2003)

$$\frac{dB(t,t,r)}{B(t,t,r)} = (r(t) + b(t,t)s_r x)dt - b(t,t)s_r dZ_r(t),$$

$$B(t,t) = 1,$$

where x is the market price of interest rate risk assumed to be constant, and

$$b(t,t) = \frac{1 - e^{-a(t-t)}}{a}.$$

The riskless asset has a price process governed by

$$dR(t) = R(t)r(t)dt,$$

$$R(0) = R_0. \qquad (2)$$

The riskless asset can be considered as a cash fund paying the instantaneous interest rate $r(t)$ without any default risk. The value of the cash fund at t is then

$$R(t) = R(0)\exp\left[\int_0^t r(s)ds\right]. \qquad (3)$$

There are zero-coupon bonds for any date of maturity, and a bond rolling over zero coupon bonds with constant maturity K. The price of the zero coupon bond with constant maturity K is denoted by $B_K(t, r)$ with

$$\frac{dB_K(t,r)}{B_K(t,r)} = [r(t) + b_K s_r x]dt - b_K s_r dZ_r(t) \,, \tag{4}$$

where

$$b_K = \frac{1 - e^{-aK}}{a} \quad .$$

The relationship between $B(t, t, r)$ and $B_K(t,r)$ through the riskless cash asset $R(t)$ (Boulier et al, 2001) is

$$\frac{dB(t,t,r)}{B(t,t,r)} = \left(1 - \frac{b(t,t)}{b_K}\right)\frac{dR(t)}{R(t)} + \frac{b(t,t)}{b_K}\frac{dB_K(t,r)}{B_K(t,r)} \,.$$

The above equation shows that the "rolling bond" can be obtained by a portfolio of one zero coupon bond and the cash asset, and that other bonds can also be obtained through a portfolio of the riskless asset and the "rolling bond".

The stock has a process of the total return (that is, the value of a single premium investment in the stock with reinvestment of dividend income) governed by stochastic differential equation (SDE)

$$dS(t) = S(t)\left[m_S(r,t)dt + v_{rS} s_r dZ_r(t) + s_S dZ_S(t)\right],$$

$$S(0) = S_0 \,, \tag{5}$$

where

$$m_S(r,t) = r(t) + s x_S \tag{6}$$

is the instantaneous percentage change in stock price per unit time. The total stock instantaneous volatility $s = \sqrt{v_{rS}{}^2 s_r{}^2 + s_S{}^2}$ is assumed to be constant, and the volatility scale factor v_{rS} measures how the interest rate volatility affects the stock volatility. The risk premium on the stock is $m_S = s x_S$, where the market price of stock risk, x_S, is assumed to be constant.

The market as assumed above has a diffusion matrix given by

$$\Sigma \equiv \begin{bmatrix} -b_K s_r & 0 \\ v_{rS} s_r & s_S \end{bmatrix}, \qquad (7)$$

and s_r and s_S are assumed to be different from zero and the diffusion matrix is invertible.

2.2. Wages

The plan member's wage, $Y(t)$, evolves according to the SDE

$$dY(t) = Y(t)\left[(m_Y(t) + r(t))dt + v_{rY} s_r dZ_r(t) + v_{SY} s_S dZ_S(t) + s_Y dZ_Y(t)\right],$$

$$Y(0) = Y_0, \qquad (8)$$

where $m_Y(t)$ is a deterministic function of time, age and other individual characteristics such as education and occupations. These assumptions on wage processes are similar to those by Battochio and Menoncin (2004) and Cairns et al (2006). Here s_Y is a constant and $Z_Y(t)$ a standard Brownian motion, independent of $Z_r(t)$ and $Z_S(t)$. The volatility scaling factors, v_{rY} and $v_{SY,}$ measure how interest rate volatility and stock volatility affect wage volatility, respectively. The parameter s_Y is a non-hedgeable volatility whose risk source does not belong to the set of the financial market risk sources. When $s_Y = 0$, the market is complete. Otherwise the market is incomplete.

2.3. The fund wealth and wealth-to-wage ratio

The value of the plan member's pension fund is denoted by $W(t)$, and the proportions of fund wealth invested in the riskless asset, bonds and stock are denoted as $q_R(t)$, $q_B(t)$ and $q_S(t)$ respectively. Since all pension wealth is invested in those three types of asset,

$$q_R(t) + q_B(t) + q_S(t) = 1, \qquad (9)$$

The change in the pension wealth (dW) at time t comes from two sources: returns from investment of pension wealth and contributions from the wage income (Y) at time t. Using $q_R(t) + q_B(t) + q_S(t) = 1$, the SDE governing the pension wealth process is

$$dW(t) = W(t)\left[(1-q_B-q_S)\frac{dR}{R}+q_B\frac{dB}{B}+q_S\frac{dS}{S}\right]+pY(t)dt$$
$$= \{W(t)[(1-q_B-q_S)r+q_B(r+b_K s_r x)+q_S m_S]+pY(t)\}dt \qquad (10)$$
$$+W(t)(-q_B b_K+q_S v_{rS})s_r dZ_r+W(t)q_S s_S dZ_S\,.$$

where p is the proportion of wage contributed to the pension plan and $Y(t)$ is the wage income at period t.

As in Cairns et al (2006), the terminal utility is assumed to be a function of terminal pension wealth-to-final wage ratio, $X(T)=W(T)/Y(T)$ and independent of the interest rate at time T, $r(T)$,

$$U(X(T),r(T)) \equiv U(X(T))\,.$$

Applying Itô 's lemma, the SDE governing the wealth-to-wage ratio is

$$dX(t) = \frac{1}{Y}dW - \frac{W}{Y^2}dY + \frac{W}{Y^3}(dY)^2 - \frac{1}{Y^2}(dWdY)\,. \qquad (11)$$

By substituting the value of W, Y, dW and dY, the SDE governing this pension wealth-to-wage ratio process is:

$$dX(t) = [(q'M+u)X+p]dt+(q'\Gamma'+\Lambda')XdZ\,, \qquad (12)$$

where,

$$q' \equiv [q_B \quad q_S],$$

$$M \equiv \begin{bmatrix} b_K s_r x + b_K v_{rY} s_r^{\,2} \\ m_S - v_{rY} v_{rS} s_r^{\,2} - v_{SY} s_S^{\,2} \end{bmatrix},$$

$$u \equiv -m_Y + v_{rY}^{\,2} s_r^{\,2} + v_{SY}^{\,2} s_S^{\,2} + s_Y^{\,2}\,,$$

$$\Gamma' \equiv \begin{bmatrix} -b_K s_r & 0 & 0 \\ v_{rS} s_r & s_S & 0 \end{bmatrix},$$

$$\Lambda' \equiv [-v_{rY} s_r \quad -v_{SY} s_S \quad -s_Y],$$

$$Z \equiv [Z_r \quad Z_S \quad Z_Y]'\,. \qquad (13)$$

The new diffusion matrix for the financial market is given by Γ. I assume that $(\Gamma'\Gamma)$ is invertible in all following sections.

3. The optimization problem and Hamilton-Jacobi-Bellman equation

I assume that the expected terminal utility has the functional form

$$V(t,x,r,y;q) = E[U(X_q(T), r(T), Y(T)) \mid X(t) = x, r(t) = r, Y(t) = y] \quad (14)$$

where $X_q(t)$ is the path of $X(t)$ given the strategy q. The optimal asset allocation problem is to find the strategy q that maximize the expected terminal utility of a plan member,

$$J(t,x,r,y) = \sup_q V(t,x,r,y;q) \quad (15)$$

The above specifications have a similar form to those in Cairns et al (2006). The stochastic optimal control problem can be written as follows:

$$\max_q E[U(X(T),T)],$$

subject to

$$d\begin{bmatrix} w \\ X \end{bmatrix} = \begin{bmatrix} m_w \\ [(q'M+u)X+p \end{bmatrix} dt + \begin{bmatrix} \Omega' \\ (q'\Gamma'+\Lambda)X \end{bmatrix} dZ,$$

$$w(0) = w_0, X(0) = X_0, \forall 0 \le t \le T, \quad (16)$$

where,

$$\underset{2\times 1}{w} \equiv [r \quad Y]',$$

$$\underset{2\times 1}{m_w} \equiv [a(b-r) \quad Y(m_Y + r)]',$$

$$\underset{2\times 3}{\Omega'} \equiv \begin{bmatrix} s_r & 0 & 0 \\ Yv_{rY}s_r & Yv_{SY}s_S & Ys_Y \end{bmatrix}. \quad (17)$$

The solution to this problem should give us the optimal portfolio composition.

The Hamiltonian corresponding to the optimization problem (16) is

$$H(J) = J_t + m'_w \frac{\partial J}{\partial w} + [(q'M+u)x + p] \frac{\partial J}{\partial X} + \frac{1}{2} tr\left(\Omega'\Omega \frac{\partial^2 J}{\partial w^2}\right)$$
$$+ (q'\Gamma'+\Lambda')\Omega x \frac{\partial^2 J}{\partial w \partial X} + \frac{1}{2}(q'\Gamma'\Gamma q + 2q'\Gamma'\Lambda + \Lambda'\Lambda)x^2 \frac{\partial^2 J}{\partial X^2}. \quad (18)$$

Differentiating equation (18) with respect to q gives the first-order condition

$$\frac{\partial H}{\partial q} = Mx\frac{\partial J}{\partial X} + \Gamma'\Omega x \frac{\partial^2 J}{\partial w \partial X} + (\Gamma'\Gamma q + \Gamma'\Lambda)x^2 \frac{\partial^2 J}{\partial X^2} = 0, \quad (19)$$

where $\frac{\partial H}{\partial q}$ is a vector. The optimal portfolio composition is

$$q^* = -(\Gamma'\Gamma)^{-1}\Gamma'\Lambda - (\Gamma'\Gamma)^{-1} M \frac{J_X}{xJ_{XX}} - (\Gamma'\Gamma)^{-1}\Gamma'\Omega \frac{J_{wX}}{xJ_{XX}}. \qquad (20)$$

where the subscripts on J indicate partial derivatives. Here $q^* = [q_B(t)^* \quad q_S(t)^*]'$, the optimal proportions invested in bonds and stock respectively. The three terms on the right hand side of equation (20) can be designated as q_1*, q_2* and q_3* respectively, which are themselves vectors with two elements corresponding to certain proportions of investment in bonds and stock. We can also view q_1*, q_2* and q_3* as three mutual funds constructed with bonds and stock.

The three terms on the right-hand-side of equation (20) correspond to the optimal asset allocation strategy with three mutual funds labeled as "cash", "bond" and "equity" in Cairns et al (2006). Their three funds are linear combinations of the three terms in equation (20). The above equation concerns with the optimal composition of the mutual fund constructed with risky assets, bonds and stocks. This is in contrast with the conclusion of Cairns et al (2006) that the first term represents a mutual fund dominated by cash assets. Since the vector q does not include the proportion of pension wealth invested in the risk free asset, the actual optimal portfolio consists of two components: one is cash assets with proportion $1 - q_B - q_S$, and the other is a mutual fund with proportion $q_B + q_S$. From this analysis, we get

Proposition 1: *The optimal portfolio consists of two components: 1) a risk free asset with proportion* $1 - q_B - q_S$*, and 2) risky assets with proportion* $q_B + q_S$*. The risky component can be further divided into three funds: a) a preference-free hedging component,* $-(\Gamma'\Gamma)^{-1}\Gamma'\Lambda$ *(fund 1), b) a speculative component,* $-(\Gamma'\Gamma)^{-1} M \frac{J_X}{xJ_{XX}}$ *(fund 2), and c) a state variable dependent hedging component,* $-(\Gamma'\Gamma)^{-1}\Gamma'\Omega \frac{J_{wX}}{xJ_{XX}}$ *(fund3).*

The preference-free component (fund 1) minimizes the instantaneous variance of the wealth-to-wage ratio differential, dX. From equation (12), the variance of dX is given by

$$\text{var}(dX) = (q'\Gamma'\Gamma q + 2q'\Gamma'\Lambda + \Lambda'\Lambda)X^2 dt .$$

Minimizing $\text{var}(dX)$ over θ gives a minimum-variance portfolio, which is identical to fund 1.

The speculative component (fund 2) increases when the "returns" on wealth-to-wage ratio $X(t)$ (i.e. M) increase, and decreases when the relative risk aversion ($-XJ_{XX}/J_X$) or the wealth-to-wage ratio variance ($\Gamma'\Gamma$) increases. Here the "returns" on wealth-to-wage ratio $X(t)$ (i.e. M) means wage adjusted returns on the assets, not the original returns from the assets. The speculative component is to satisfy the risk appetite of the plan members.

The state variable dependent component (fund 3) depends explicitly on the diffusion terms of the state variables (Ω), suggesting that this component covers the plan member from financial market risk. In fact, the present formulation uses the member's wage as a numeraire to assess the fund manager's performance.

4. Optimal asset allocation strategy for power terminal utility

Non-linear partial differential equations such as those derived in the preceding section generally do not have closed-form solutions, although closed-form solutions may be found for some particular form of utility functions. To compute the optimal portfolio composition θ for power utility, it is necessary to find the maximum expected utility function $J(t,x,w)$, which should satisfy the boundary condition $J(T, x, w) = \frac{1}{1-g} x^{1-g}$, where g is the relative risk aversion coefficient. Based on whether there are pension contributions from wage incomes and whether there is a non-hedgeable wage risk, the optimal allocation strategy for power utility can be considered in three scenarios: 1) there is no contribution from future wage incomes ($p = 0$) (with or without non-hedgeable risk; the two cases have the same solution in the present model); 2) there are pension contributions from wage incomes ($p > 0$), but there is no non-hedgeable wage risk ($s_Y = 0$); 3) there are pension contributions from wage incomes ($p > 0$), and there is a non-hedgeable wage risk ($s_Y \neq 0$). Since there is no analytical solution for the third scenario ($p > 0$, $s_Y \neq 0$), in the present study I will only work on the first two scenarios.

4.1. Optimal asset allocation without wage income contribution, p =0

Since $p = 0$, equation (18) (the Hamilton-Jacobi-Bellman equation) becomes

$$H(J) = J_t + m'_w \frac{\partial J}{\partial w} + (q'M + u)x\frac{\partial J}{\partial X} + \frac{1}{2}tr\left(\Omega'\Omega\frac{\partial^2 J}{\partial w^2}\right)$$
$$+ (q'\Gamma' + \Lambda')\Omega x \frac{\partial^2 J}{\partial w \partial X} + \frac{1}{2}(q'\Gamma'\Gamma q + 2q'\Gamma'\Lambda + \Lambda'\Lambda)x^2 \frac{\partial^2 J}{\partial X^2} \ . \qquad (21)$$

I start with a trial solution by assuming that the maximized expected terminal utility of plan members has the functional form

$$J(t,x,w) = \frac{1}{1-g} g(t,w)^g x^{1-g} \ , \qquad (22)$$

$$g(T,w) = 1 \qquad \forall w \, .$$

Then, we have

$$J_t = \frac{g}{1-g} g^{g-1} g_t x^{1-g} \ ,$$

$$J_x = g^g x^{-g} \ ,$$

$$J_{xx} = -g g^g x^{-g-1} ,$$

$$J_w = \frac{g}{1-g} g^{g-1} g_w x^{1-g} \ ,$$

$$J_{ww} = -g g^{g-2} g_w{}^2 x^{1-g} + \frac{g}{1-g} g^{g-1} g_{ww} x^{1-g} \ ,$$

$$J_{xw} = g g^{g-1} g_w x^{-g} \ . \qquad (23)$$

In the above equations J_w, J_{xw} and g_w are vectors, and J_{ww} and g_{ww} are matrices. Substituting the partial derivatives of the expected terminal power utility function in (23) into the above HJB equation (21) gives

$$\begin{aligned} &\frac{g}{1-g} g^{g-1} g_t x^{1-g} + (q'M + u) g^g x^{1-g} + m_w' \frac{g}{1-g} g^{g-1} g_w x^{1-g} \\ &+ \frac{1}{2} tr\left\{\Omega'\Omega\left[(-g) g^{g-2} g_w{}^2 x^{1-g} + \frac{g}{1-g} g^{g-1} g_{ww} x^{1-g}\right]\right\} \\ &+ (q'\Gamma' + \Lambda')\Omega g g^{g-1} g_w x^{1-g} + \frac{1}{2}[q'\Gamma'\Gamma q + 2q'\Gamma'\Lambda + \Lambda'\Lambda](-g) g^g x^{1-g} = 0. \end{aligned} \qquad (24)$$

Substituting the optimal proportion composition of pension fund investment θ*, equation (20), and simplifying,

$$\begin{aligned} &g_t + \left[m_w' + \frac{1-g}{g} M'(\Gamma'\Gamma)^{-1}\Gamma'\Omega\right] g_w + \frac{1}{2} tr(\Omega'\Omega g_{ww}) \\ &- \left[\frac{1-g}{2(-g)^2} M'(\Gamma'\Gamma)^{-1} M + \frac{1-g}{g} M'(\Gamma'\Gamma^{-1})\Gamma'\Lambda - \frac{1-g}{g} u\right] g = 0. \end{aligned} \qquad (25)$$

For the $p = 0$ scenario, obviously the function $g(t,w)$ has to satisfy equation (25) for the assumption $J(t,x,w) = \frac{1}{1-g} g(t,w)^g x^{1-g}$ to be a correct solution. By the Feynman-Kac formula (Øksendal 2000; Duffie 2001), there exists a probability measure $Q(g)$ such that

$$g(t, w(t)) = E_{Q(g)}[g(T, \tilde{w}(T)) D(t,T) \mid F_t], \qquad (26)$$

where $\tilde{w}(s)$ is governed by the SDE

$$d\tilde{w}(s) = \tilde{m}_w(\tilde{w}(s))ds + \Omega(\tilde{w}(s), s)'dZ,$$

$$\tilde{m}_w(\tilde{w}(s)) = m_w + \frac{1-g}{g} M'(\Gamma'\Gamma)^{-1}\Gamma'\Omega,$$

$$\tilde{w}(t) = w(t),$$

and

$$D(t,T) = \exp\left[\int_t^T j(s)ds\right],$$

where

$$j\,(s) = -\left[\frac{1-g}{2(-g)^2} M'(\Gamma'\Gamma)^{-1} M + \frac{1-g}{g} M'(\Gamma'\Gamma)^{-1}\Gamma'\Lambda - \frac{1-g}{g} u \right].$$

In equation (26), F_t is the filtration, which can be interpreted as the information available to the investor at time *t*. The *s* in $j\,(s)$ stands for time, and the function is written as $j\,(s)$ to indicate that j might be a function of time (if one or more of the parameters in M, Γ, Λ and *u* are time dependent). In the present study, all parameters in M, Γ, Λ and u are assumed to be constant, and therefore $j\,(s)$ are constant.

Using the results from the Feynman-Kac formula, i.e. (26), the optimal portfolio composition (equation (20)) is

$$q^* = -(\Gamma'\Gamma)^{-1}\Gamma'\Lambda - (\Gamma'\Gamma)^{-1} M \frac{1}{-g} + (\Gamma'\Gamma)^{-1}\Gamma'\Omega \int_t^T \frac{\partial}{\partial w_t} E_t[j\,(s)]ds\,. \qquad (27)$$

Since all the terms in the function *j (s)* do not depend on the state variables *r* and *Y*, its derivatives with respect to w_t are zero and the above equation becomes

$$q^* = -(\Gamma'\Gamma)^{-1}\Gamma'\Lambda + (\Gamma'\Gamma)^{-1} M \frac{1}{g}. \qquad (28)$$

In equation (28), only the second term, i.e. the speculative component, depends on the relative risk aversion *g*. The first term in the above equation is

$$q_1^* = -(\Gamma'\Gamma)^{-1}\Gamma'\Lambda$$
$$= \frac{-1}{b_K{}^2 s_r{}^2 s_S{}^2} \begin{bmatrix} b_K v_{rY} s_r{}^2 s_S{}^2 - b_K v_{rS} v_{SY} s_r{}^2 s_S{}^2 \\ -b_K{}^2 v_{SY} s_r{}^2 s_S{}^2 \end{bmatrix} = \frac{1}{b_K} \begin{bmatrix} v_{rS} v_{SY} - v_{rY} \\ b_K v_{SY} \end{bmatrix};$$

The second term is

$$q_2^* = (\Gamma'\Gamma)^{-1} M \frac{1}{g}$$
$$= \frac{1}{g b_K s_r s_S{}^2} \begin{bmatrix} v_{rS}{}^2 s_r{}^2 x + s_S{}^2 x + v_{rY} s_r s_S{}^2 + v_{rS} m_S s_r - v_{rS} v_{SY} s_r s_S{}^2 \\ b_K (v_{rS} s_r{}^2 x + m_S s_r - v_{SY} s_r s_S{}^2) \end{bmatrix}.$$

The optimal proportions of pension wealth invested in bonds and equities are

$$\begin{bmatrix} q_B{}^* \\ q_S{}^* \end{bmatrix} = \frac{1}{b_K} \begin{bmatrix} v_{rS} v_{SY} - v_{rY} \\ b_K v_{SY} \end{bmatrix}$$
$$+ \frac{1}{g b_K s_r s_S{}^2} \begin{bmatrix} v_{rS}{}^2 s_r{}^2 x + s_S{}^2 x + v_{rY} s_r s_S{}^2 + v_{rS} m_S s_r - v_{rS} v_{SY} s_r s_S{}^2 \\ b_K (v_{rS} s_r{}^2 x + m_S s_r - v_{SY} s_r s_S{}^2) \end{bmatrix}. \qquad (29)$$

The optimal proportion of pension wealth invested in risk-free assets can be calculated by using

$$q_R^* = 1 - q_B^* - q_S^* . \tag{30}$$

Since there is no time or horizon dependent variable in those equations, we get

Theorem 1: *Under the market formulation and optimization objective specified in this paper, when there is no further contribution from wage incomes, the optimal portfolio composition for power utility is horizon and time independent.*

It is clear that the optimal proportions of pension wealth invested in the three asset categories are not time-dependent. Although I have used pension wealth-to-wage ratio rather than wealth as the argument of power utility and there is no consumption-derived utility in my formulation, I get the same conclusion as those by Samuelson (1969) and Merton (1969, 1971). The optimal asset allocation is time independent.

Theorem 1 is based on results from the π=0 scenario and in the next subsection I will show that it is also true for the $p \neq 0$ and $s_Y = 0$ scenario. Because there is no analytical solution for the $p \neq 0$ and $s_Y \neq 0$ scenario, it is not clear for the time being whether Theorem 1 applies to the $p \neq 0$ and $s_Y \neq 0$ scenario. The optimal asset allocation for the $p \neq 0$ and $s_Y \neq 0$ scenario can be solved numerically, but the numerical solution would be very complicated with the present model formulation. Cairns et al (2006) in their paper solving for the π=0 scenario and $s_Y = 0$ scenarios commented, the $p \neq 0$ and $s_Y \neq 0$ scenarios "involve a type of computational analysis that is sufficiently different and sufficiently extended to justify a separate paper".

The present results demonstrate that when the terminal utility is a power function of wealth-to-wage ratio, the state variables dependent hedging component (the third term of equation (20)) disappears. This result is consistent with those in

Cairns et al (2006). The state variables dependent hedging component corresponds to the "bond" fund in Cairns et al (2006) and they conclude that "bond" fund becomes zero when the pension plan is funding for a cash lump sum (more precisely it should be when funding for wealth-to-wage ratio, because "bond" fund does not become zero when funding for a cash lump sum).

The preference free hedging component (the first term of equation (20)) is to hedge wage risk, corresponding to the "cash" fund in Cairns et al (2006). In the sense that it contains bonds and stocks but not cash assets, the present result is different from that of Cairns et al. The difference between the present study and Cairns et al (2006) in terms of "cash" fund may be more in appearance than in substance. When they concluded that "The optimal weight in risky assets is equivalent to investing in three mutual funds denoted A, B and C. Fund A …… will be dominated by cash", they did not indicate whether $p_A = (p_{A1},..., p_{AN})'$ or $p_A = (p_{A0},..., p_{AN})'$ is dominated by cash asset, where $p_{A0} = 1 - \sum_{i=1}^{N} p_{Ai}$ is the proportion invested in the riskless asset. Since $p_A = (p_{A1},..., p_{AN})'$ is investment in risky assets, it cannot be dominated by cash assets. The correct interpretation of their results is that $p_A = (p_{A0},..., p_{AN})'$ is dominated by cash asset.

Cairns et al (2006) denote the "cash" fund asset proportion vector as p_A with $p_{A0} = 1 - \sum_{i=1}^{N} p_{Ai}$, and similarly for "bond" fund p_B and "equity" fund p_C, effectively splitting the investment in the riskless asset to the three mutual funds. In this way, the dominance by cash assets in "cash" fund is dominance by p_{A0} in p_A, with p_{A0} the proportion in cash assets. There is no dominance by cash assets in the wage risk hedging $p_A = (p_{A1},..., p_{AN})'$, which is used in the derivation of optimal composition, because there is no cash asset.

Expressing the optimal portfolio composition in the same way as that of Cairns et al (2006), we have

$$q^* = q_A p_A + q_D p_D + q_C p_C,$$

where $q_A = 1 - \frac{J_{wX}}{xJ_{XX}} + \frac{J_X}{xJ_{XX}}$, $q_D = \frac{J_{wX}}{xJ_{XX}}$, and $q_C = 1 - q_A - q_B = -\frac{J_X}{xJ_{XX}}$ with $p_A = -(\Gamma'\Gamma)^{-1}\Gamma'\Lambda$, $p_D = -(\Gamma'\Gamma)^{-1}\Gamma'(\Lambda - \Omega)$, and $p_C = -(\Gamma'\Gamma)^{-1}\Gamma'(\Lambda - M)$. From those expressions it can be seen that the products of three mutual funds and their proportions with respect to pension wealth are linear combinations of the three components in equation (20). The proportions of cash assets in the three mutual funds are $p_{A0} = 1 - 1'[-(\Gamma'\Gamma)^{-1}\Gamma'\Lambda]$, $p_{D0} = 1 - 1'[-(\Gamma'\Gamma)^{-1}\Gamma'(\Lambda - \Omega)]$, and $p_{C0} = 1 - 1'[-(\Gamma'\Gamma)^{-1}\Gamma'(\Lambda - M)]$, with $1' = [1 \quad 1]$. Because $\frac{J_{wX}}{xJ_{XX}} = 0$, the optimal asset allocation strategy is to invest in two mutual funds A with asset proportion vector p_A ($p_{A0} = 1 - 1'[-(\Gamma'\Gamma)^{-1}\Gamma'\Lambda]$) and C with asset proportion vector p_C ($p_{C0} = 1 - 1'[-(\Gamma'\Gamma)^{-1}\Gamma'\Lambda]$). The weights in the two mutual funds are $q_A = 1 + \frac{J_X}{xJ_{XX}}$ and $q_C = 1 - q_A = -\frac{J_X}{xJ_{XX}}$, respectively.

Since the preference free hedging component is to hedge wage risk, it becomes zero when the correlations between interest rate and wage growth and between stock return and wage growth are zero. This result is consistent with the finding of Cairns et al (2006) that the "cash" fund contains 100% cash assets when wage growth and asset returns are uncorrelated.

The speculative component (the second term of equation (20)), however, seems not so consistent with the conclusion of Cairns et al (2006) that the "equity" fund is dominated by equities. In the present study using wealth-to-wage ratio, the second term of equation (20) contains a substantial investment in bonds, suggesting that the "equity" fund include substantial amount of bonds. In my numerical results later, the speculative component or "equity" fund is actually dominated by bonds. If the present result applies to the scenario of one riskless and N risky assets, the "equity" fund in Cairns et al (2006) cannot be literally understood as consisting of only or even mainly equities.

Table 2-1 Parameters used in numerical simulation

Interest rate	Value
Mean reversion, a,	0.2
Mean rate, b	0.05
Volatility, s_r	0.02
Initial rate, r_0	0.05
Fixed maturity bond	
Maturity, K	20 years
Market price of risk, x	0.15
Stock	
Risk Premium, m_S	0.06
Stock own volatility, s_S	0.19
Interest volatility scale factor, v_{rS}	1 or -1
Wage	
Wage premium, m_Y	0.01
Non-hedgeable volatility, s_Y	0.01
Interest volatility scale factor, v_{rY}	0.7
Stock volatility scale factor, v_{SY}	0.9
Initial wage, Y_0	10k
Contribution rate, p	10%
Length of pension plan, T	45

To illustrate the above solution for the asset allocation problem, I have calculated the proportions of different assets for different values of relative risk aversion with the parameters in Table 1. The values of parameters are those commonly used in other pension studies to facilitate comparison (Boulier et al 2001; Deelstra et al 2003; Cairns et al 2006; Battocchio and Menoncin 2004). The results are shown in Fig.1. It is easy to see from Fig.1 that with the parameters commonly used, stock is a very safe asset and an individual will stop short-selling cash for buying stock only when her relative risk aversion is high ($g > 5.15$ if $v_{rS} = -1$ and $g > 45.45$ if $v_{rS} = 1$ in my calculation).

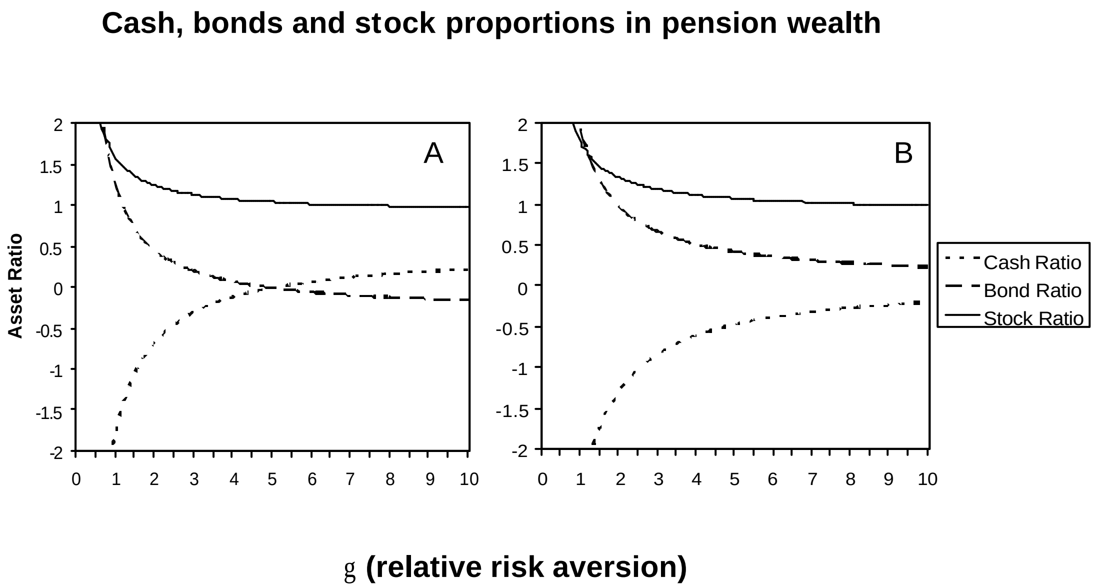

Fig. 1 The relationship between relative risk aversion coefficient g and the optimal proportions of cash, bond and stock invested for pension wealth. The asset ratio range is cut off at -2 and 2 in order to show details of asset proportion when $g > 1$. A. Interest volatility scale factor for stock, $v_{rS} = -1$. B. $v_{rS} = 1$.

The preference-free component (the first term in equation (20)) is dominated by the stock, and the speculative component (the second term in equation (20)) is dominated by the bonds, which does not support the conclusion on "equity" fund by Cairns et al (2006). An increase in the relative risk aversion coefficient γ reduces the short-sale of cash assets and consequently the (absolute) proportions invested in both bonds and stocks. Because γ only affects the speculative component which is dominated by bonds with commonly assumed market parameters, the overall effect of an increase in γ reduces the proportion invested in bonds relative to that invested in stocks. At about $g = 0.7$ if $v_{rS} = -1$ and $g = 1.2$ if $v_{rS} = 1$, the investment in stock overtakes that in bonds. The relative risk aversion has often been estimated in the range between 2 and 4. In this range, there is a short-sale of cash assets between 12.7% and 68% if $v_{rS} = -1$ and between 60% and 130% if $v_{rS} = 1$. As shown in Fig.2, when the relative risk aversion gets smaller, the short-sale of cash asset can be many times the value of pension fund.

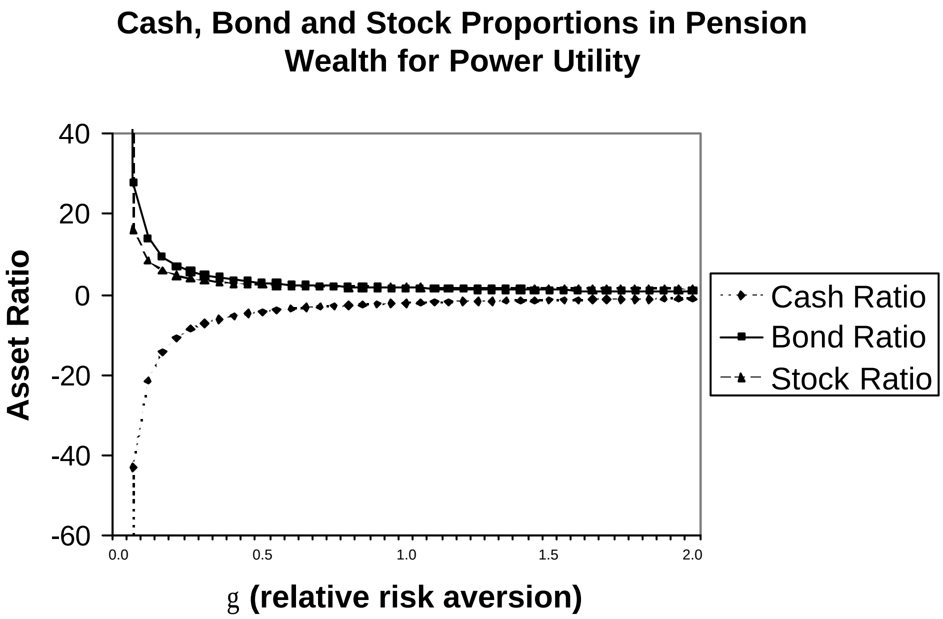

Fig.2 The relationship between g and the optimal proportions of cash, bond and stock invested for $v_{rS} = 1$. The g range is cut off at 2 in order to show the full extent of cash short-sale and purchase of bond and stock.

As can be seen in Fig. 1, if $v_{rS} = -1$ and $g < 4.7$, it is optimal to short-sell bonds in order to hold more stocks. The economic intuition here is that with $v_{rS} < 0$ the changes in bond and stock prices caused by innovations in the spontaneous interest rate are in the same direction, whereas the changes in the risk free asset and the stock are in the opposite directions. Therefore, long in cash and short in bonds can hedge against changes in stock. The reason why this only happens with relatively high g is the higher return of bonds compared with cash assets. The more risk tolerant investors prefer the higher return of bonds to the safety of risk free assets. The estimates of v_{rS} are quite variable in different pension strategy studies, ranging from -5 (Cairns et al 2006) to 3 (Battocchio and Menoncin 2004). Deelstra et al (2003) use a value of 0.02 and Boulier et al (2001) use -3. In the present thesis I use middle range values of $v_{rS} = -1$ and $v_{rS} = 1$.

The present numerical results indicate that for individuals with $g > 2$, substantial more pension wealth should be invested in stocks than in bonds. These results are qualitatively consistent with the asset distributions of pension funds in UK. As at 31 March 2006, the average asset distribution is 35.8% in UK equities, 28.9% in overseas equities, 23.1% in bonds, 7.6% in index-linked gilts, 2.4% in property, 1.8% in cash, and 0.4% in other assets, demonstrating the dominance of equities (Mellon Analytical Solutions: UK Pension Fund Analysis to 31 March 2006). Quantitatively, the average asset distribution in terms of the ratio between bonds and equities corresponds to a g of around 2 if $v_{rS} = -1$ and around 6 if $v_{rS} = 1$, roughly within the range of usual g estimates. Two points should be noted in such comparisons. One is that the pension funds are generally not allowed to short-sell cash assets, and we do not place a short-sale constraint on the optimal asset allocation. The other is the "tension in economics between the attempt to describe the optimal choices of fully rational individuals ('positive economics') and the desire to use our models to improve people's imperfect choices ('normative economics')" as commented by Campbell and Viceira (2002). My present results, like most portfolio strategy studies, show that optimal asset allocation strategies for power utility are usually horizon independent, but financial advisors often recommend a horizon dependent lifestyle strategy.

My calculations also showed that volatilities, s_r and s_S, have a profound influence on the optimal proportions of the three assets. A reduction in the stock volatility increases the proportion of wealth invested in the stock and decreases the proportion invested in the cash asset, while the proportion invested in the bond has a smaller increase. A reduction in interest rate volatility increases the proportion invested in the bond and decreases the proportion invested in the cash asset, whereas the proportion invested in the stock also decreases. An increase in the stock risk premium m_S increases the proportion invested in the stock and decreases the proportion invested in the cash assets, whereas there is only a marginal increase in the proportion invested in the bond.

4.2. Optimal asset allocation with hedgeable wage income contribution

When wage income is fully hedgeable ($s_Y = 0$), u, Γ, Λ and Z in equation (12), the SDE governing the wealth-to-wage ratio process, become:

$$u \equiv -(m_Y - v_{rY}^2 s_r^2 - v_{sY}^2 s_S^2),$$

$$\Gamma' \equiv \begin{bmatrix} -b_K s_r & 0 \\ v_{rS} s_r & s_S \end{bmatrix},$$

$$\Lambda \equiv -[v_{rY} s_r \quad v_{sY} s_S]',$$

$$Z' \equiv [Z_r \quad Z_S]', \qquad (31)$$

The diffusing term of the state variables is

$$\underset{2\times 2}{\Omega'} \equiv \begin{bmatrix} s_r & 0 \\ Y v_{rY} s_r & Y v_{SY} s_S \end{bmatrix}. \qquad (32)$$

While in the scenario without further contribution from wage incomes the initial wealth has to be assumed, in the scenario of fully hedgeable wages the initial wealth can be calculated from wages and market parameters. To borrow against future wage income it is necessary to calculate the market value at time t for future premiums payable between t and T. The wage, $Y(t)$, evolves according to the SDE

$$dY(t) = Y(t)\left[(m_Y(t) + r(t))dt + v_{rY}s_r dZ_r(t) + v_{SY}s_S dZ_S(t)\right],$$

$$Y(0) = Y_0, \qquad (33)$$

Let Q be the risk-neutral pricing measure and $\tilde{Z}_r(t)$ and $\tilde{Z}_S(t)$ independent standard Q-Brownian motions (Cairns et al 2006), the wage process under Q is

$$dY(t) = Y(t)\left[(m_Y(t) + r(t) - x_r v_{rY}s_r - x_S v_{SY}s_S)dt + v_{rY}s_r d\tilde{Z}_r(t) + v_{SY}s_S d\tilde{Z}_S(t)\right], \qquad (34)$$

which implies that

$$Y(t) = Y(t)\exp\Big\{\int_t^t [m_Y(s) + r(s)]ds - (x_r v_{rY}s_r + x_S v_{SY}s_S + \frac{1}{2}v_{rY}{}^2 s_r{}^2 + \frac{1}{2}v_{SY}{}^2 s_S{}^2)(t - t)$$
$$+ v_{rY}s_r[\tilde{Z}_r(t) - \tilde{Z}_r(t)] + v_{SY}s_S[\tilde{Z}_S(t) - \tilde{Z}_S(t)]\Big\}. \qquad (35)$$

Here x_r is a measure of how interest/bond volatility will affect wage, and x_S is a scale factor measuring how stock price volatility affects wages. They are essentially prices of risks. The market value at time t for future premiums payable between t and T is then

$$E_Q\left[\int_t^T \exp\left\{-\int_t^t r(s)ds\right\}pY(t)dt \mid F_t\right]$$
$$= pE_Q\left[\int_t^T Y(t)\exp\left\{\int_t^t m_Y(s)ds - (x_r v_{rY}s_r + x_S v_{SY}s_S + \frac{1}{2}v_{rY}{}^2 s_r{}^2 + \frac{1}{2}v_{SY}{}^2 s_S{}^2)(t - t)\right.\right.$$
$$\left.\left. + v_{rY}s_r[\tilde{Z}_r(t) - \tilde{Z}_r(t)] + v_{SY}s_S[\tilde{Z}_S(t) - \tilde{Z}_S(t)]\right\}dt \mid F_t\right]$$
$$= pY(t)\int_t^T \exp\left\{\int_t^t m_Y(s)ds - (x_r v_{rY}s_r + x_S v_{SY}s_S)(t - t)\right\}dt$$
$$= pY(t)f(t). \qquad (36)$$

The pension plan can have an additional wealth of $pY(t)f(t)$ by short-selling a replicating portfolio of value $-pY(t)f(t)$, which will be paid off exactly by future contributions from wage incomes. The total pension wealth enhanced with the present market value of future contributions is $W(t) + pY(t)f(t)$, and the current enhanced pension wealth-to-wage ratio is $X(t) + pf(t)$. The optimal composition of pension fund will be the same as in the case of no wage income contribution, but the optimal terminal utility function will have the form

$$J(t,x,w) = \frac{1}{1-g} g(t,w)^g (x + pf(t))^{1-g} . \tag{37}$$

The optimal strategy is to hold $-pY(t)f(t)$ in the replicating portfolio and invest the $W(t) + pY(t)f(t)$ in the optimal composition of pension fund wealth. Such a treatment of hedgeable wage income risk is often applied in portfolio and pension studies (Deestra et al 2003; Cairns et al 2006). The composition of the replicating portfolio can be written in vector form

$$q^R = \begin{bmatrix} q_B{}^R \\ q_S{}^R \\ q_R{}^R \end{bmatrix} = \begin{bmatrix} \dfrac{v_{rS} v_{SY} - v_{rY}}{b_K} \\ v_{SY} \\ 1 - v_{SY} - \dfrac{v_{rS} v_{SY} - v_{rY}}{b_K} \end{bmatrix} . \tag{38}$$

In the above equation, the superscript R indicates replicating portfolio.

The sum $\tilde{W}(t) = W(t) + pY(t)f(t)$ can be denoted as the augmented pension wealth and $\tilde{X}(t) = X(t) + pf(t)$ augmented wealth-to-wage ratio. The total pension financial wealth is the sum of the invested augmented pension wealth $\tilde{W}(t)$ and the short-sold replicating portfolio $-pY(t)f(t)$. The matrix representation of the wealth-to-wage ratio SDE and the HJB equation for the augmented wealth-to-wage ratio is the same as that when there is no further income contribution, although the parameters u, Γ, Λ, Z and Ω in equations (12) and (16) are replaced by those defined in equations (31) and (32). The optimal portfolio composition when expressed in matrix form is the same as that without further contributions from wage incomes in the preceding subsection.

When u, Γ, Λ, Z and Ω in equations (31) and (32) are substituted into the optimal solution, equation (27) or (28), the optimal proportions of pension wealth invested in bonds and equities are

$$\begin{bmatrix} q_B * \\ q_S * \end{bmatrix} = \frac{1}{b_K}\begin{bmatrix} v_{rS}v_{SY} - v_{rY} \\ b_K v_{SY} \end{bmatrix}$$
$$+\frac{1}{gb_K s_r s_S{}^2}\begin{bmatrix} v_{rS}{}^2 s_r{}^2 x + s_S{}^2 x + v_{rY} s_r s_S{}^2 + v_{rS} m_S s_r - v_{rS} v_{SY} s_r s_S{}^2 \\ b_K (v_{rS} s_r{}^2 x + m_S s_r - v_{SY} s_r s_S{}^2) \end{bmatrix}.$$

These expressions for optimal proportions in bonds and stocks are the same as those for the no further contribution ($p = 0$) case. The optimal proportion of pension wealth invested in risk-free assets can be calculated by using

$$q_R(t)^* = 1 - q_B(t)^* - q_S(t)^* .$$

From the above analysis, we get

Proposition 2: *When there is no nonhedgeable wage risk, the optimal pension portfolio composition is the same as that when there is no further contribution from wage incomes.*

Since the net value of the pension plan financial wealth is *W(t)*, the optimal composition of financial wealth is

$$\begin{bmatrix} q_B{}^F \\ q_S{}^F \end{bmatrix} = \left(1 + \frac{pY(t) f(t)}{W(t)}\right)\begin{bmatrix} q_B * \\ q_S * \end{bmatrix} - \frac{pY(t) f(t)}{W(t)}\begin{bmatrix} q_B{}^R \\ q_S{}^R \end{bmatrix}$$
$$= \left(1 + \frac{pY(t) f(t)}{W(t)}\right)\begin{bmatrix} \dfrac{v_{rS}v_{SY} - v_{rY}}{b_K} \\ v_{SY} \end{bmatrix}$$
$$+\left(1 + \frac{pY(t) f(t)}{W(t)}\right)\begin{bmatrix} \dfrac{v_{rS}{}^2 s_r{}^2 x + s_S{}^2 x + v_{rY} s_r s_S{}^2 + v_{rS} m_S s_r - v_{rS} v_{SY} s_r s_S{}^2}{gb_K s_r s_S{}^2} \\ \dfrac{v_{rS} s_r x + m_S - v_{SY} s_S{}^2}{g s_S{}^2} \end{bmatrix},$$
$$-\frac{pY(t) f(t)}{W(t)}\begin{bmatrix} \dfrac{v_{rS}v_{SY} - v_{rY}}{b_K} \\ v_{SY} \end{bmatrix}$$

which can be simplified as

$$\begin{bmatrix} q_B{}^F \\ q_S{}^F \end{bmatrix} = \begin{bmatrix} \dfrac{v_{rS}v_{SY}-v_{rY}}{b_K} \\ v_{SY} \end{bmatrix} + \left(1+\frac{pY(t)f(t)}{W(t)}\right)\begin{bmatrix} \dfrac{v_{rS}{}^2 s_r{}^2 x + s_S{}^2 x + v_{rY} s_r s_S{}^2 + v_{rS} m_S s_r - v_{rS} v_{SY} s_r s_S{}^2}{g b_K s_r s_S{}^2} \\ \dfrac{v_{rS} s_r x + m_S - v_{SY} s_S{}^2}{g s_S{}^2} \end{bmatrix}. \qquad (40)$$

In the above equation, $q_B{}^F$ and $q_S{}^F$ are the optimal proportions of financial wealth invested in bonds and stocks respectively, and $q_B{}^R$ and $q_S{}^R$ are proportions of the replicating portfolio short-sold in bonds and stocks respectively. The optimal proportion of the financial wealth invested in risk-free assets can be calculated by using

$$q_R(t)^F = 1 - q_B(t)^F - q_S(t)^F .$$

The value of replicating portfolio decreases as *t* increase (i.e. the retirement date approaches), whereas *W(t)* is generally increasing in *t*. Because the optimal composition of the augmented pension wealth is different from the composition of the replicating portfolio, the change in their relative sizes will affect the optimal composition of their sum, the financial wealth. Therefore, although neither the optimal composition of augmented pension wealth nor the composition of the replicating portfolio is time or horizon dependent, the optimal composition of the pension plan financial wealth is horizon-dependent. Equation (40) shows that the composition of pension plan financial wealth is horizon dependent.

From the above analysis, we get

Proposition 3: *When there is no nonhedgeable wage risk, while the optimal portfolio composition of the augmented pension wealth is the same as that of pension wealth when there is no further contribution from wage incomes, the optimal composition of* *the pension plan financial wealth (the augmented pension wealth + short-sold replicating portfolio) is horizon dependent.*

The scenario with contributions from hedgeable wage incomes differs from that with no further contributions in that initial wealth can be calculated and borrowed against future wage incomes (by short-selling the replicating portfolio) and in that the composition of financial wealth is horizon-dependent. The scenario of no further contribution can be considered as a special case of hedgeable wage incomes where $pY(t)f(t) = 0$. As illustrated in Fig. 3 where parameters in Table 1 and γ=2 are used in the numerical simulation, the optimal proportions of the three assets are horizon dependent. The values of cash, bond and stock in the financial wealth and the total value of financial wealth (in terms of wealth-to-wage ratio) over the life of the pension plan are shown in Fig.4.

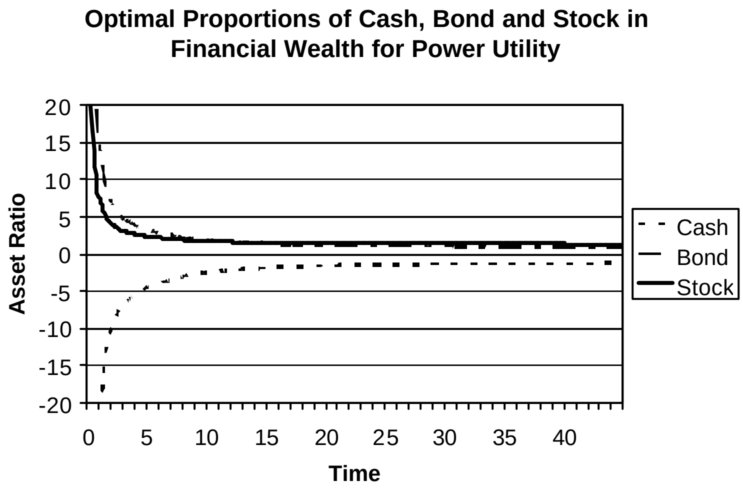

Fig.3 The horizon-dependent profile of optimal proportions of cash, bond and stock in financial wealth for power utility. Parameters in Table 1 are used in the simulation and the relative risk aversion coefficient γ=2. The results are from 100 simulations.

Since the net value of the financial wealth (augmented wealth+short position in the replicating portfolio) is small at early stage (it is zero at t=0), the composition of the financial wealth has a large short position in cash assets to finance the long position in bonds and stocks. As the short position in the replicating portfolio is paid off gradually by future contributions, the net value of the financial wealth increases and the asset ratio (the ratio between the wealth invested in one class of assets and the financial wealth) of all the three assets decrease. At later stage of the pension plan, since the short-sold replicating portfolio becomes very small compared with the augmented pension wealth or the net value of the financial wealth, the optimal composition of the financial wealth is very similar to that of the augmented wealth. At the end of the pension plan, the optimal composition and the net value of the financial wealth are identical to those of the augmented pension wealth.

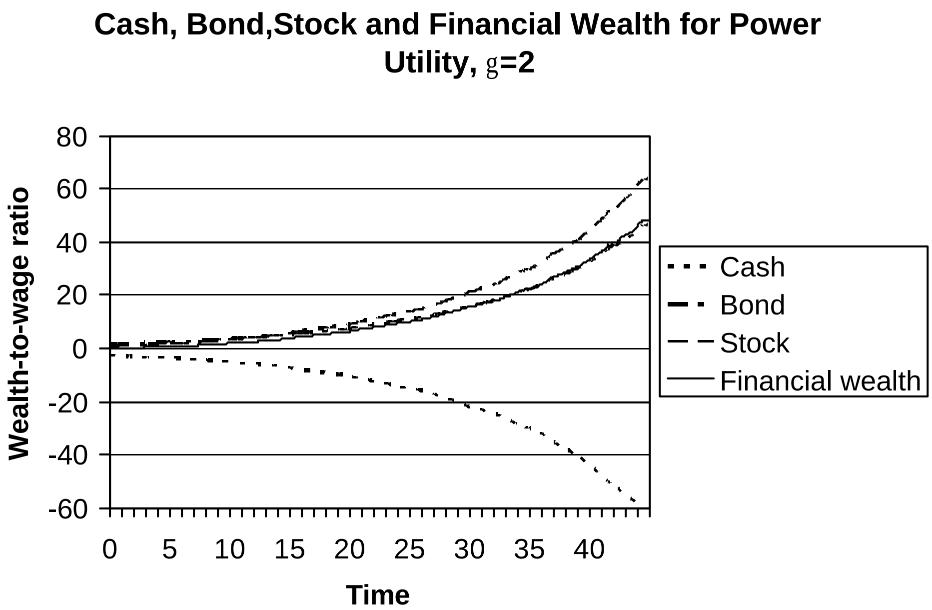

Fig.4 The values of cash, bond and stock in the financial wealth and the total value of financial wealth (in terms of wealth-to-wage ratio) over the life of the pension plan. Parameters in Table 1 are used in the simulation and the relative risk aversion coefficient γ=2. The results are from 100 simulations.

The above results indicate that when there is stochastic wage income and the wage income risk is hedgeable, the optimal composition of the pension plan financial wealth is horizon-dependent, although the optimal asset allocation of the augmented pension wealth is horizon or time-independent. Short-selling of risk-free asset is considered as reduction in the holding of risk free asset, and the proportion of risky assets in the financial wealth (sum of the augmented pension wealth and the replicating portfolio) is higher than that when there is no contribution from wage incomes. The short-sold replicating portfolio is being paid off over time, so that the proportion of riskless asset in the financial wealth increases and the proportions of risky assets decrease. The optimal portfolio composition in terms of financial wealth is therefore stochastic lifestyling (Cairns et al 2006). This is consistent with the results of Bodie et al (1992) and Campbell and Viceira (2002) that the presence of (risky) labor incomes tilts the portfolios towards risky financial assets.

Concerning the horizon-dependence of optimal portfolio composition, there is a subtle difference between the present results and those from Boulier et al (2001), Deelstra et al (2003) Battocchio and Menoncin (2004) and Cairns et al (2006). In the studies of Boulier et al (2001), Deelstra et al (2003) Battocchio and Menoncin (2004) and Cairns et al (2006), the optimal composition of pension wealth portfolio per se is horizon dependent when interest rates are stochastic, while in the present study the optimal composition of pension wealth portfolio per se is horizon independent and it is the optimal composition of financial wealth that is horizon dependent. The assumption that the expected terminal utility is a function of wealth-to-wage ratio may underlie the difference. Boulier et al (2001) and Deelstra et al (2000, 2003) have considered pension plans with a guaranteed minimum benefit at retirement and the terminal utility measured as a power function of surplus cash over the guaranteed benefit. Using wealth-to-wage ratio, as the argument of the expected terminal utility, in the present study removes the dependence of instantaneous conditional expected change per unit time (the expression multiplying with *dt* in the SDE) on the state variables, so that the need to hedge against the fluctuations in the state variables disappears. The assumption by Cairns et al (2006) that terminal utility is a function of replacement ratio re-introduces the dependence of instantaneous conditional expected change per unit time on the state variables, because replacement ratio is the quotient between wealth-to-wage ratio and annuity price and annuity price is interest rate dependent.

5. Conclusion

In this paper I have solved the optimal portfolio problem under stochastic interest rate and wage income for power utility, using three assets cash, bonds and stock. I assume that the terminal utility of a pension plan member is a function of terminal pension wealth-to-wage ratio. Under the present model assumptions, the optimal portfolio (for an unspecified utility function) invests in both riskless and risky assets. The investment in risky assets contains three components: a preference free hedging component to hedge wage risk, a speculative component proportional to both portfolio Sharpe ratio and the inverse of the Arrow-Pratt relative risk aversion index, and a state variable dependent hedging component to hedge financial market risks. This result is consistent with that of Cairns et al (2006). The three components are roughly corresponding to the "cash", "equity" and "bond" funds in Cairns et al (2006).

Closed form solution is derived for power terminal utility when there is no further contribution from wage incomes or when there is no non-hedgeable wage risk. The state-variable dependent hedging component disappears with the assumption that the expected terminal utility is a power function of wealth-to-wage ratio. The preference free hedging component and the speculative component contain both bonds and stocks, which is different from the conclusion of Cairns et al (2006) that the "cash fund" is dominated by cash assets and the "equity" fund is dominated by stocks. I find that even the speculative component ("equity" fund) can have a larger proportion of bonds with commonly assumed market parameters. Since both the preference free hedging component and the speculative component are horizon independent, the optimal pension asset allocation strategy of pension wealth per se is horizon independent. When the future contributions from wage incomes are hedged by short-selling a replicating portfolio, the optimal portfolio composition of pension plan financial wealth (augmented pension wealth + short-sold wage replicating portfolio) is horizon dependent.

To summarize, the optimal asset allocation of pension wealth portfolio for DC pension plan members with terminal utility as a power function of pension wealth-to-wage ratio, which invests in both riskless and risky assets, is horizon independent. The optimal portfolio composition of pension plan financial wealth is horizon dependent. The investment in risky assets contains a preference free component to hedge wage risk and a speculative component to satisfy the risk appetite of the plan members. The two components consist of both bonds and stocks.

Acknowledgements

This paper is part of the author's PhD thesis. The author is grateful to Prof. D. Blake, Dr. S. Wright, Dr. B. Baxter, Dr. D. McCarthy and Dr. Khorasanee for their helpful comments and constructive suggestions.

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