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An investment based valuation approach for pension fund cash flows

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Abstract

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* The views expressed in this paper are those of the author, and may not be shared by the Pension Fund City of Zurich.

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Applying modern asset pricing theory, a new valuation approach for pension fund liabilities is proposed. One of its important features is that it creates strong incentives for the fund manager to hold an asset portfolio with a high Sharpe ratio and good hedging characteristics against the fund's cash flow. Such a portfolio should provide high financial security to the fund at low economic costs. Furthermore, it is shown that discounting pension liabilities at the risk free rate or the expected asset return are both valid only in special cases.

1 Introduction

The fair or market valuation of pension fund cash flows is a hotly debated topic and has recently become standard in international accounting (e.g. IAS 19). In Switzerland, however, pension fund accounting standards have not yet been changed to fair valuation of liabilities. Rather, a fix technical interest rate – 4 % for many Swiss pension funds – is used to discount the fund's liabilities. With bond yields now for years well below this rate and in light of the increased volatility of the funding ratio due to the newly introduced market valuation of the asset side, the discussion on fair valuation of liability has started. The usual way is to assume the pension fund's expected cash flow to be deterministic and discount it at the (nearly) risk less interest rate. The goal of this paper is to propose a more general pricing framework which encompasses this standard approach as a special case and which sets desirable incentives to the fund's asset management.

Market valuation of pension liabilities in its narrow sense is out of scope in most cases, as there are almost no traded assets which depend on the risk factors driving pension fund cash flows, namely salary growth rates and rates on mortality and morbidity. Therefore, a model based approach has to be taken. The simplest model discounts the pension fund's expected cash flow at the risk less interest rate. Although this approach is easy to understand and implement, it has severe drawbacks. Firstly, it is often too conservative as it does not account for the hedging capabilities of available financial instruments. Secondly, this approach sets strong incentives to invest the fund's assets heavily in bonds. However, such an investment strategy may have undesirable effets:

- With given contribution rates, the investment strategy might be infeasible in the sense that its returns may be too low to finance defined benefits. Thus, heavy bond investments may lead to an increase in contribution rates.
- As the bond market – at least for long-term maturities – is too narrow to cover all pension fund liabilities, such an investment strategy followed by many pension funds leads to low interest rates as is currently the case in the United Kingdom (see NEUE ZÜRCHER ZEITUNG [2006a,b]). This in turn increases the value of the liabilities and further reduces investment returns, starting a vicious circle.

The problems inherent in the standard valuation approach are caused by treating pension fund cash flows as deterministic which obviously is not the case. They depend at least on factors like salary growth rates, mortality and morbidity rates and – in the case of defined contribution plans – also on savings. All these factors are not known in advance with certainty rendering the fund's cash flow stochastic. The more general approach put forward in this paper recognizes the cash flow's inherent stochasticity and is based on the theoretically well-founded stochastic discount factor methodology.

A pension fund's primary duty is to secure the financing of its liabilities at the lowest economic costs, i.e. with the lowest possible contribution rates for the active workforce and the employer. Ideally, the fund's investment strategy tries to match its future cash flows as well as possible and to deliver as high returns as possible. As such, the fund's investment strategy should use all instruments available in the financial market which provide some hedge against its cash flow. Based on this approximate replication argument it is proposed to define the stochastic discount factor, which prices the fund's cash flow, as a function of its investment strategy.

This valuation approach has several favourable characteristics. It properly accounts for the random factors in a pension fund's cash flow and uses as much market information on them as possible. Furthermore, the approach strongly encourages the fund to invest its assets in a well-diversified portfolio and at the same time to hedge its cash flow as well as possible. Investing in a well-diversified portfolio circumvents the above mentioned problems of heavy bond investments. The incentive to hedge the cash flow as well as possible reduces surplus volatility and may also reduce the investment strategy's dependence on the possibly different views of the fund's stakeholders (the employer, the active workforce, the retirees, the board of trustees, consultants, the auditor and the employer's shareholders).

The remainder of the paper is organized as follows: The next section briefly describes basic characteristics of a pension fund cash flow. Section three is a brief introduction to the stochastic discount factor methodology. Its application to a pension fund cash flow and the main results of the paper are found in section four. Section five gives some guidelines for an implementation and conclusions are drawn in section six.

2 Pension fund cash flows

Pension fund valuation is based on assuming a closed population, i.e. turnover in the active workforce is not considered. Cash outflows for leaving and cash inflows from entering employees are therefore not accounted for. This is reasonable as future employees have not as yet any claim against the fund.[1]

Actuarial valuation methods consider future benefit payouts only. Benefits are either attributed to specific years of service and in valuing the fund's liabilities only those benefits accumulated up to the valuation date are considered[2]; or, total future benefits are estimated and spread over the years of employment according to some rule[3]. From an economic point of view, however, it is sensible to base a pension fund valuation on its future net cash in- and outflows just as is done in company valuation.[4] A pension fund's cash inflows consist of contribution payments by the active workforce and the employer; cash outflows include benefit payments to retirees, disabled, widows and orphans and the fund's operating costs.

A stochastic version of the approach developed by METTLER (2005) is used to model a pension fund's net cash flow. For a final salary pension scheme, cash in- and outflows are a function of the population structure, employees' salaries and contribution and benefit rates.[5] The stochastic net cash flow $\tilde{c}_t$ at time t can be characterized by the following, slightly simplified model (the tilde indicates stochastic variables):

$$(1) \qquad \tilde{c}_t = P_0\left(\prod_{\tau=1}^{t}\tilde{\Phi}_\tau\right)\tilde{L}_t S_0 \prod_{\tau=1}^{t}\left(1+\tilde{g}_\tau\right)$$

The fund's population is divided into z groups each of which is characterized by age and status such as active, retired, disabled, widowed or dead. The z-dimensional row vector P_0 describes the population as at the valuation date. The population is assumed to follow a discrete semi-Markov process with stochastic transition matrix $\tilde{\Phi}_\tau \in \Re^{z \times z}$. The initial salary structure is given by the z-dimensional column vector S_0. Salary growth is governed by a stochastic growth rate $\tilde{g}_\tau$ reflecting price inflation and productivity growth. For expositional simplicity, it is assumed first that there is no salary career and second that benefit payments are adjusted to wage inflation. Indeed, at least for public employers in Switzerland there is ample evidence that wage inflation follows consumer price inflation very closely. $\tilde{L}_t \in \Re^{z \times z}$ links the population with the salary structure for each point in time. It contains the contribution and benefit rates as percentages of the current salaries applicable for each possible state transition of the population (for details, see METTLER [2005]). With equation (1), it is easy to compute the fund's net cash flow $\tilde{c}_t$ at each future point in time t.

3 The stochastic discount factor methodology

It is useful to briefly review the stochastic discount factor methodology.[6] For ease of exposition, a simple two period model with discrete states s, s=1,…,S, describing uncertainty is used.[7] It is assumed that for each state there exists a basis asset which pays one franc in this state and zero in all other states. The prices of these basis assets are denoted as $p_{s,t}$ and are called state prices. Given a new asset with state-dependent payoff $h_{s,t+1}$, it's price is easily discovered as

(2) $$p_t = \sum_{s=1}^{S} p_{s,t} h_{s,t+1} .$$

This is a relative pricing approach, as market information in form of the state prices is used to value the stochastic cash flow $h_{s,t+1}$. However, the basis assets need not be traded by themselves; it is sufficient if enough non-redundant assets are traded from which the state prices can be extracted. The state prices comprise the following information:

- the probability $\pi_{s,t+1}$ of state s to realize at time t+1;
- the investor's time preference, i.e. the time-value of money;
- and the investor's risk assessment of state s, i.e. the marginal utility that is attached to an additional franc obtained in state s.

Dividing the state prices by their respective probabilities one arrives at the state deflators or stochastic discount factors

(3) $$m_{s,t+1} = \frac{p_{s,t}}{\pi_{s,t+1}} .$$

This directly leads to the general asset pricing equation

$$(4) \qquad p_t = \sum_{s=1}^{S} \pi_{s,t+1} m_{s,t+1} h_{s,t+1} = E_t\left[m_{t+1} h_{t+1}\right]$$

E_t is the expectation operator conditioned on information available at time t. Given the role the state deflator m_{t+1} plays in this pricing equation, it is called stochastic discount factor. As every discount factor, it accounts for the time-value of money. In addition, it also reflects the investor's risk assessment of the different states which constitutes its stochastic nature. The more unfavourable a state is considered to be, the higher is the corresponding discount factor reflecting that an additional franc is very welcome in an unfavourable state (e.g. a recession).

The asset pricing equation (4) holds in complete and incomplete markets. In complete markets, a unique solution for the stochastic discount factor can be obtained from observed market prices. An incomplete market setting, on the other hand, is characterized by incomplete information on m_{t+1} resulting in pricing bounds instead of a unique price p_t.

In order to give some more structure on equation (4), a risk free asset is introduced which – by definition – pays a franc in each state and the price p_t^f of which is obtained by applying pricing equation (4):

$$(5) \qquad p_t^f \equiv \frac{1}{R_{t+1}^f} = \sum_{s=1}^{S} \pi_{s,t+1} m_{s,t+1} 1 = E_t\left[m_{t+1}\right]$$

with R_{t+1}^f denoting the gross risk free rate for the maturity (t+1) which is known at time t. The first equation in (5) is true by definition. Thus, the expected value of the stochastic discount factor equals the price of the risk less asset. Applying the covariance rule to equation (4) and inserting (5) results in

$$(6) \qquad \begin{aligned} p_t &= E_t\left[m_{t+1}\right] E_t\left[h_{t+1}\right] + Cov_t\left[m_{t+1}, h_{t+1}\right] \\ &= \frac{E_t\left[h_{t+1}\right]}{R_{t+1}^f} + Cov_t\left[m_{t+1}, h_{t+1}\right] \end{aligned}$$

The first term on the right hand side of the second equation is the usual risk free discounting of the expected cash flow. The second term is a risk-adjustment which is determined by how strong the cash flow h_{t+1} co varies with the stochastic discount factor. A positive covariance means that the cash flow is often high when the discount factor is high, i.e. when it is badly needed. This implies a high price for the cash flow which is equivalent to a low expected return. In turn, a negative covariance leads to a low price and a high expected return which is necessary to convince investors to buy the relatively unattractive cash flow which pays a lot in good times and little in bad times. It is only if the cash flow has zero covariance with the discount factor that the instrument's price equals the risk less discounted expected cash flow. This is the case if the instrument's cash flow is deterministic or if no market information on the risk factors driving the cash flow is available.

Pricing equation (6) must hold for any two periods and therefore extends straightforwardly to the multi-period case where a cash flow arrives at T points in time:

(7) $$p_t = \sum_{j=1}^{T} \frac{E_t\left[h_{t+j}\right]}{R_{t+j}^f} + Cov_t\left[m_{t+j}, h_{t+j}\right]$$

The risk free rate R_{t+j}^f is the yield of a risk less bond maturing in (t+j) periods.

4 Valuation of pension fund cash flows

A pension fund's stochastic cash flow $\tilde{c}_t$ as given in equation (1) can be valued with equation (7) as soon as the dynamics of the stochastic discount factors are specified. In a complete market, where state prices for all possible states can be extracted from traded instruments, this is an easy task. Alas, there are far too few instruments traded which reveal information about the states relevant for a pension fund as e.g. mortality rates, salary growth rates and so on.

Thus, only pricing bounds can be obtained from the available, incomplete market information (see e.g. CAIRNS et. al. (2006) for an application). When it comes to accounting, however, pricing bounds are of little value. To pin down a single value for the fund's cash flow, some strong assumption has to be made. In the standard approach, it is assumed that the expected cash flow is deterministic; in this setting, all relevant pricing information is easily obtained from traded bonds. In order not to give up the stochastic nature of the pension fund cash flow, it is alternatively suggested to assume that the possible states of the fund's asset returns cover the whole uncertainty about the fund's cash flows. Put differently, it is assumed that all information needed to price the fund's cash flow can be extracted from the returns of the fund's assets. This allows to obtain s unique price for the fund's stochastic cash flow.

For analytical tractability, the stochastic discount factor is modelled as a linear function of the fund's random gross asset return $\tilde{R}_t^A = 1 + \tilde{r}_t^a$:

(8) $$\tilde{m}_t = a_t - b_t \tilde{R}_t^A$$

where a_t and b_t are some constants given information at time t. The linear form of equation (8) is in line with important asset pricing models such as the Capital Asset Pricing Model (CAPM) or the Arbitrage Pricing Theory (APT) which are all based on similar linear forms for the stochastic discount factor.

As argued in the introduction, the rationale behind equation (8) is that in pursuing financial stability, a fund should hedge its cash flow as well as possible with available traded assets. If this is the case, the fund's investment strategy comprises as much pricing information on its cash flow as possible. If a pension fund actually behaved in this ideal way, it seemed reasonable to attach a single value to its cash flow based on the fund's asset return. As will be shown below, the pricing model emerging from this line of reasoning precisely sets the incentives for the fund to manage its assets as it ideally should. Thus, if a pension fund for any reason excludes assets which would hedge its cash flow will discount the cash flow at too high a rate.

Using the definition of the stochastic discount factor in equation (8) in conjunction with the pricing equation (7), the fair value v_t of a pension fund's cash flow is given by

$$
\text{(9)} \qquad \begin{aligned} v_t &= \sum_{j=1}^{T} \frac{E_t\left[\tilde{c}_{t+j}\right]}{R^f_{t+j} + \dfrac{E_t\left[\tilde{R}^A_{t+1}\right] - R^f_{t+j}}{\sigma_t\left[\tilde{R}^A_{t+1}\right]}\sigma_t\left[\tilde{R}^c_{t+1}\right]\rho_t\left[\tilde{R}^A_{t+1}, \tilde{R}^c_{t+1}\right]}, \\ &= \sum_{j=1}^{T} \frac{E_t\left[\tilde{c}_{t+j}\right]}{R^f_{t+j} + SR^A_{t+1}\sigma_t\left[\tilde{R}^c_{t+1}\right]\rho_t\left[\tilde{R}^A_{t+1}, \tilde{R}^c_{t+1}\right]} \end{aligned}
$$

where $\tilde{R}^c_{t+j} = \dfrac{\tilde{c}_{t+j}}{v_{t+j-1}}$ is the gross growth rate of the fund’s cash flow, SR^A_{t+1} is the investment strategy’s conditional Sharpe ratio and σ_t and ρ_t denote the conditional standard deviation and correlation respectively (the derivation of equation (9) is given in the appendix).

The stochastic rate used to discount the fund’s cash flow thus depends on the risk free interest rate, which reflects the time-value of money, and a risk-adjustment which is determined by the Sharpe ratio of the fund’s investment strategy, the risk inherent in the fund’s cash flow and the hedge effect of the fund’s assets as measured by their correlation with the cash flow. With positive correlations, the risk-adjustment is positive and the discount rate lies between the risk free rate and the expected asset return.

From equation (9), it becomes obvious that discounting at the risk free rate or at the expected asset return will be inappropriate in most cases:

- Discounting the fund’s expected cash flow at the risk free rate is valid only if at least one of the following conditions hold:
    - the fund’s assets are fully invested into the risk less asset such that the Sharpe ratio becomes zero;
    - the fund’s cash flow has zero volatility, i.e. it is deterministic;
    - the fund’s asset return is completely uncorrelated with its cash flow implying that either there is no hedge effect or the pension fund is risk neutral.

  The first point is easy to check and needs no further discussion. As to the second point, pension fund cash flows never show zero conditional volatility. Future payments to the current beneficiaries are probably the least volatile cash flows; still they are stochastic due to the probability of the beneficiaries to change their status and to possible adjustments to inflation. Regarding the last point, this is basically an empirical question. However, it should be save to assume at least a small hedging effect of available financial instruments like inflation-linked bonds, real estate or stocks. Thus, in most cases, discounting at the risk free rate is too conservative.
- It is generally inappropriate to discount the fund’s expected cash flow at the expected asset return which would be the easiest way to perform an investment based valuation of the fund’s cash flow.[8] This would only be valid if the investment strategy provided a perfect hedge implying $\rho_t\left[\tilde{R}^A_{t+1}, \tilde{R}^c_{t+1}\right] = 1$ and $\sigma_t\left[\tilde{R}^A_{t+1}\right] = \sigma_t\left[\tilde{R}^c_{t+1}\right]$. If this is not the case, doing so results in an overly optimistic discounting which neglects investment and cash flow risks.

More importantly, the pricing model (9) sets desirable incentives to the fund management:

- There is an incentive for the fund to invest its assets into a well-diversified portfolio resulting in a high Sharpe ratio as such a strategy lowers the fair value of its cash flow. The rationale is that the higher the investment strategy’s risk-adjusted return (Sharpe ratio), the easier it is to finance the fund’s liabilities; thus, a lower fair value can be attached to the cash flow. Another important point is that the investment strategy's risk is explicitly taken into account. It is impossible for the fund to increase expected asset returns without increasing its volatility and vice versa, unless it has, up to now, followed a mean-variance-inefficient investment strategy.[9]

- The fund can lower the fair value of its cash flow by implementing with its assets a good a hedge for its cash flow as possible such that $\rho_t\left[\tilde{R}_{t+1}^{A}, \tilde{R}_{t+1}^{c}\right]$ gets as close to one as possible and the asset returns' volatility corresponds closely to that of the cash flow growth rate. The fund management has thus a strong incentive to use all available and possibly new instruments which provide some hedge against its liabilities. If a higher investment than cash flow risk is chosen, resulting in a comparably high expected excess return, only part of the assets' excess return goes into the discount rate, as in this case $\sigma_t\left(\tilde{R}_t^c\right) \big/ \sigma_t\left(\tilde{R}_t^A\right) < 1$. Thus, the level of expected excess return going into the discount rate is determined solely by the fund' cash flow risk $\sigma_t\left(\tilde{R}_t^c\right)$.

There will usually be a trade off between a high Sharpe ratio and a good cash flow hedge as there is no reason to assume the market portfolio – which offers the maximum Sharpe ratio – provides the best possible hedge against the cash flow. This implies that pension funds often will invest into mean-variance-inefficient portfolios. This, however, need not be of any concern as neither quadratic utility nor normally distributed returns are assumed. Without such restrictive assumptions, any portfolio within the upper and lower bound of the mean-variance efficient frontier is eligible (see e.g. COCHRANE [2001]).

5 Implementation

The implementation of the pricing model (9) is obviously more involved than discounting at the risk free rate. Equation (9) is a conditional pricing model where expected returns and second moments may be time-varying. An integrated model describing the dynamics of assets and liabilities is needed. There are several approaches put forward in the literature to model multivariate asset returns (see e.g. WILKIE [1986], CAMPBELL, CHAN and VICEIRA [2003] or SCHERER [2003]). As for the liability part, the approach of METTLER (2005) may be used. In integrating the asset and liability models it is important that both are driven by the same underlying macroeconomic risk factors such as inflation and economic growth.

Simulating such an integrated model, the conditional moments in equation (9) can be obtained for a specific time t by taking empirical moments across all simulated paths at time step t. This simulation based approach to estimating conditional moments was introduced into the literature by LONGSTAFF and SCHWARTZ (2001) in an option pricing application and was later used e.g. by BRANDT et. al. (2005) in dynamic optimization.

6 Conclusions

The standard approach to fair valuation of pension fund cash flows is to discount the expected cash flow at a risk less interest rate. It is shown here that this approach disregards the stochastic nature of a pension fund's cash flow and results in a too conservative valuation. Moreover, pension funds are pushed to heavy bond investments in order to reduce surplus volatility leading to lower expected asset returns and higher contribution rates than necessary.

Using a linear function of the fund's asset return as stochastic discount factor, a valuation formula which is deeply rooted in modern asset pricing theory and which delivers economically sensible results is found. The discount rate lies between the risk free rate and the investment strategy's expected return depending on how good the hedge against the fund's cash flow provided by its investments is. The valuation approach creates incentives for the fund manager to invest into a well-diversified portfolio with a high Sharpe ratio and with good hedging characteristics. Such a portfolio will ensure as much financial stability to the fund at as low costs to the active workforce and the employer as is possible given current financial markets.

Appendix

In this appendix, the two-period counterpart of the pricing equation (9) is derived in detail. To start, the general asset pricing formula (4) is stated in return form, which is obtained by dividing (4) by the price p_t:

$$\text{(A1)} \quad 1 = E_t\left[\tilde{m}_{t+1}, \tilde{R}^A_{t+1}\right]$$

where $\tilde{R}^A_{t+1} = \frac{\tilde{c}_{t+1}}{p_t}$ denotes the asset's gross return. The conditional constants a_t and b_t in equation (8) can be determined by pricing any two financial instruments, e.g. the fund's assets return $\tilde{R}^A_t$ itself and the gross risk free rate $R^f_t = 1 + r^f_t$ with (A1):

$$\text{(A2)} \quad \begin{aligned} 1 &= E_t\left[\tilde{m}_{t+1}\tilde{R}^A_{t+1}\right] \\ 1 &= E_t\left[\tilde{m}_{t+1}\right]R^f_{t+1} \end{aligned}$$

Solving these two equations for a_t and b_t results in

$$\text{(A3)} \quad \begin{aligned} a_t &= \frac{1}{R^f_{t+1}} + b_t E_t\left[\tilde{R}^A_{t+1}\right] \\ b_t &= \frac{E_t\left[\tilde{R}^A_{t+1}\right] - R^f_{t+1}}{R^f_{t+1}\sigma^2_t\left[\tilde{R}^A_{t+1}\right]} \end{aligned}$$

a_t and b_t are time-varying if the risk free rate and/or the asset return's first and second moments are time-varying. As the expected asset return should be greater than the risk free rate, both a_t and b_t are positive. With equation (8) this means that a high asset return, signalling good times, implies a low stochastic discount factor.

Now, pricing the pension fund's cash flow gross growth rate $\tilde{R}^c_t$ by (A1) and applying the covariance rule yields

$$
\begin{aligned}
1 &= E_t\left[\tilde{m}_{t+1}\tilde{R}^c_{t+1}\right] \\
&= E_t\left[\tilde{m}_{t+1}\right]E_t\left[\tilde{R}^c_{t+1}\right] + \mathrm{cov}_t\left[\tilde{m}_{t+1},\tilde{R}^c_{t+1}\right] \\
&= \frac{E_t\left[\tilde{R}^c_{t+1}\right]}{R^f_{t+1}} + \mathrm{cov}_t\left[\tilde{m}_{t+1},\tilde{R}^c_{t+1}\right]
\end{aligned} \tag{A4}
$$

The covariance term in (A4) is obtained by using again the covariance rule and inserting (8) for the stochastic discount factor:

$$
\begin{aligned}
\mathrm{cov}_t\left[\tilde{m}_{t+1},\tilde{R}^c_{t+1}\right] &= E_t\left[\tilde{m}_{t+1}\tilde{R}^c_{t+1}\right] - E_t\left[\tilde{m}_{t+1}\right]E_t\left[\tilde{R}^c_{t+1}\right] \\
&= E_t\left[\left(a_t - b_t\tilde{R}^A_{t+1}\right)\tilde{R}^c_{t+1}\right] - \frac{E_t\left[\tilde{R}^c_{t+1}\right]}{R^f_{t+1}} \\
&= a_t E_t\left[\tilde{R}^c_{t+1}\right] - b_t E_t\left[\tilde{R}^A_{t+1}\tilde{R}^c_{t+1}\right] - \frac{E_t\left[\tilde{R}^c_{t+1}\right]}{R^f_{t+1}} \\
&= \frac{E_t\left[\tilde{R}^c_{t+1}\right]}{R^f_{t+1}} + b_t E_t\left[\tilde{R}^A_{t+1}\right]E_t\left[\tilde{R}^c_{t+1}\right] - b_t E_t\left[\tilde{R}^A_{t+1}\tilde{R}^c_{t+1}\right] - \frac{E_t\left[\tilde{R}^c_{t+1}\right]}{R^f_{t+1}} \\
&= -b_t\left(E_t\left[\tilde{R}^A_{t+1}\tilde{R}^c_{t+1}\right] - E_t\left[\tilde{R}^A_{t+1}\right]E_t\left[\tilde{R}^c_{t+1}\right]\right) \\
&= -b_t\,\mathrm{cov}_t\left[\tilde{R}^A_{t+1},\tilde{R}^c_{t+1}\right]
\end{aligned} \tag{A5}
$$

Inserting (A5) into (A4) results in

$$
\begin{aligned}
1 &= \frac{E_t\left[\tilde{R}^c_{t+1}\right]}{R^f_{t+1}} - b_t\,\mathrm{cov}_t\left[\tilde{R}^A_{t+1},\tilde{R}^c_{t+1}\right] \\
&= \frac{E_t\left[\tilde{R}^c_{t+1}\right]}{R^f_{t+1}} - b_t\sigma_t\left[\tilde{R}^A_{t+1}\right]\sigma_t\left[\tilde{R}^c_{t+1}\right]\rho_t\left[\tilde{R}^A_{t+1},\tilde{R}^c_{t+1}\right]
\end{aligned} \tag{A6}
$$

Solving for the expected liability growth rate and inserting for b_t from (A3) yields

$$
\begin{aligned}
E_t\left[\tilde{R}^c_{t+1}\right] &= \left(1 + b_t\sigma_t\left[\tilde{R}^A_{t+1}\right]\sigma_t\left[\tilde{R}^c_{t+1}\right]\rho_t\left[\tilde{R}^A_{t+1},\tilde{R}^c_{t+1}\right]\right)R^f_{t+1} \\
&= R^f_{t+1} + \frac{E_t\left[\tilde{R}^A_{t+1}\right] - R^f_{t+1}}{R^f_{t+1}\sigma^2_t\left[\tilde{R}^A_{t+1}\right]}\sigma_t\left[\tilde{R}^A_{t+1}\right]\sigma_t\left[\tilde{R}^c_{t+1}\right]\rho_t\left[\tilde{R}^A_{t+1},\tilde{R}^c_{t+1}\right]R^f_{t+1} \\
&= R^f_{t+1} + \frac{E_t\left[\tilde{R}^A_{t+1}\right] - R^f_{t+1}}{\sigma_t\left[\tilde{R}^A_{t+1}\right]}\sigma_t\left[\tilde{R}^c_{t+1}\right]\rho_t\left[\tilde{R}^A_{t+1},\tilde{R}^c_{t+1}\right]
\end{aligned} \tag{A7}
$$

This is the basic pricing formula for pension fund cash flows. The expected cash flow gross growth rate or discount factor lies between the risk free rate (if the cash flow is deterministic or uncorrelated to the asset portfolio) and the expected asset return (if the asset portfolio perfectly hedges the cash flow).

Defining $\beta_t \equiv \frac{\sigma_t\left[\tilde{R}^A_{t+1}\right]\sigma_t\left[\tilde{R}^c_{t+1}\right]\rho_t\left[\tilde{R}^A_{t+1},\tilde{R}^c_{t+1}\right]}{\sigma^2_t\left[\tilde{R}^A_{t+1}\right]} = \frac{\operatorname{cov}_t\left[\tilde{R}^A_{t+1},\tilde{R}^c_{t+1}\right]}{\sigma^2_t\left[\tilde{R}^A_{t+1}\right]}$ in the usual manner as a regression coefficient, (A5) obviously looks very similar to the conditional Capital Asset Pricing Model (CAPM):

$$\text{(A8)} \quad E_t\left[\tilde{R}^c_{t+1}\right] - R^f_{t+1} = \beta_t\left(E_t\left[\tilde{R}^A_{t+1}\right] - R^f_{t+1}\right)$$

However, it is *not* the CAPM, as $\tilde{R}^A_{t+1}$ is not the market portfolio as argued in the main text.

From (A6), it is straightforward to arrive at the pricing equation (9). Multiplying (A6) through with the cash flow's fair value v_t one obtains:

$$v_t = \frac{E_t\left[\tilde{c}_{t+1}\right]}{R^f_{t+1}} - v_t b_t \sigma_t\left[\tilde{R}^A_{t+1}\right]\sigma_t\left[\tilde{R}^c_{t+1}\right]\rho_t\left[\tilde{R}^A_{t+1},\tilde{R}^c_{t+1}\right]$$

or

$$\text{(A9)} \quad v_t = \frac{E_t\left[\tilde{c}_{t+1}\right]}{R^f_{t+1}\left(1 + b_t\sigma_t\left[\tilde{R}^A_{t+1}\right]\sigma_t\left[\tilde{R}^c_{t+1}\right]\rho_t\left[\tilde{R}^A_{t+1},\tilde{R}^c_{t+1}\right]\right)}$$

$$= \frac{E_t\left[\tilde{c}_{t+1}\right]}{R^f_{t+1} + \frac{E_t\left[\tilde{R}^A_{t+1}\right] - R^f_{t+1}}{\sigma_t\left[\tilde{R}^A_{t+1}\right]}\sigma_t\left[\tilde{R}^c_{t+1}\right]\rho_t\left[\tilde{R}^A_{t+1},\tilde{R}^c_{t+1}\right]}$$

with $\tilde{c}_{t+1} = \tilde{R}^c_{t+1} v_t$.

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[1] However, it might be interesting to consider staff turnover for risk management purposes.

[2] These methods are called accrued benefit valuation methods, among which the projected unit credit method is the most popular. For a very brief introduction see e.g. KUSSMAUL and SCHWINGER (2003).

[3] These methods are called projected benefit valuation methods (see e.g. KUSSMAUL and SCHWINGER [2003]).

[4] A similar argument can be found e.g. in WARING (2004).

[5] For a treatment of defined contribution plans or hybrid schemes the reader is reffered to METTLER (2005).

[6] For an in-depth treatment the reader is referred to COCHRANE (2001) or ZIMMERMANN (1998).

[7] In this section, stochastic variables are characterized by a subscript s indicating their state; in the rest of the paper, stochastic variables are marked with a tilde.

[8] This approach was used e.g. in the United Kingdom (UK) under the old accounting standard SSAP 24 which was, however, replaced by the new FRS 17 in the year 2000.

[9] Specifically, a higher equity portion is only favourable if it leads to a higher Sharpe ratio. This was not the case in the UK under SSAP 24 (see endnote 7), rather, a higher equity portion led to a higher expected asset return which in turn lowered the liability value.