DISCUSSION PAPER PI-1909

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July 2019

ISSN 1367-580X

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On the Projection of Mortality Rates to Extreme Old Age

Kevin Dowd* and David Blake♦
This draft: 30 July 2019

Abstract

This article shows how cohort mortality rate projections of mortality models that involve age effects can be improved and extended to extreme old ages. The proposed approach allows insurers to use such mortality models to obtain valuations of financial instruments such as annuities that depend on projections of extreme old age mortality rates.

Key Words: mortality rates, Cairns-Blake-Dowd mortality model, CBDX mortality model, Lee-Carter mortality model, projection, extreme old age.

JEL codes: G220, G230, J110

1. Introduction

A common problem in life insurance is to project mortality rates out to extreme old age. This problem arises, for example, when an insurer wishes to price a life annuity. Unfortunately, a number of mortality models cannot project extreme old age mortality rates. This problem arises in mortality models of the Lee-Carter family (see Lee and Carter, 1992) which have an age-related state variable or age effect. The maximum age in the sample age range then constrains the maximum age for which one can project the corresponding mortality rates.

The exceptions are models of the Cairns-Blake-Dowd (CBD) family (Cairns *et al*., 2006, 2009). Because these models have no age state variable (SV), they can be used to project mortality rates (known as q rates) to any ages without being constrained by the range of ages in the sample data used to calibrate the age effects. Moreover, the original CBD model was designed specifically for higher ages. Currie (2011) shows how CBD can be projected to very old ages.

But what if one wants to use other models – specifically, models with age-effects – to project to very old ages? An answer is to smooth and then project the age effects, and then treat those smoothed and projected age effects as proxies for the very old age age effect that we are lacking. Ways to smooth and project these age effects have been proposed by Haberman and Renshaw (2009) and by Dowd *et alia* (2019). We can then use these projected age effects to project the q rates to any ages we wish.

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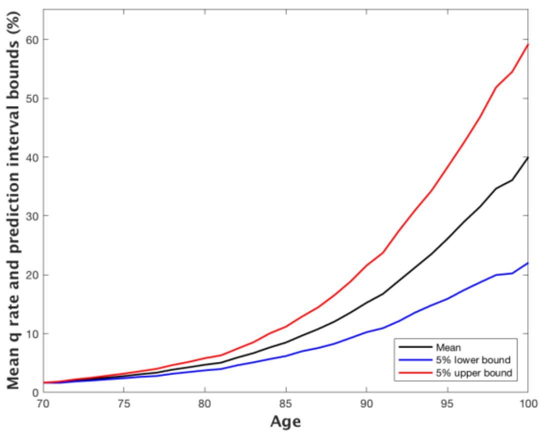
We work with the following model, known as CBDX, which combines features of both the Lee-Carter and CBD families of models.[1] This model postulates that $logm(t,x)$, the log of the death rate, is given by:

$$logm(t,x) = \alpha(x) + \sum_{i=1}^{K} \beta_i(x)\kappa_i(t) + \gamma(c) \quad (1)$$

where t refers to the time period, x refers to age, and $c = t - x$ refers to the year of birth, $\alpha(x)$, $\kappa(t)$ and $\gamma(c)$ are the age-related, period-related and cohort-related SVs respectively, and the parameters $\beta_1 = 1$, $\beta_2 = (x - \bar{x})$, $\beta_3 = (x - \bar{x})^2 - \sigma_x^2$ are fixed throughout, where $\bar{x}$ and σ_x^2 are the mean and variance of the ages in our sample age range. The difference between (1) and the original CBD M7 model is that $logm(t,x)$ replaces $logit\ q(t,x)$ and there is now a static base mortality table $\alpha(x)$.

We now use this model to obtain the q rate projections in Figure 1, where $q = 1 - e^m$. The figure shows the projected cohort q rates for an individual just turned 70, alongside the bounds of the 95% prediction intervals for the same cohort q rates. By cohort q rate, we mean that the projections follow the cohort of just-turned 70 year olds as they age over time. The projections have broadly the shape we would expect: they rise exponentially over time.

Figure 1: Projected Mean and 90% Prediction Intervals for Cohort q Rates for Australian Males Just Turned Aged 70



Notes: Projected mean and 90% prediction intervals for cohort q rates are obtained from 10,000 stochastic simulation trials based on the CBDX3 model applied to Australian male deaths and exposures data for sample years 1921:2014 and sample ages 40:100. Source: Human Mortality Database https://www.mortality.org/hmd/AUS/DOCS/ref.pdf.

However, it is apparent that the projections show a dip in the 3 last years, and so the projections at the 30-year horizon are below what we would have expected them to be had the projections from (most of) the earlier years continued out at the same rates of growth. This dip reflects increasing sample variation in the age SV as it moves into the extreme old age range (i.e., the increasing the randomness of death rates as the number of survivors decreases).

[1] This model was proposed in Dowd, Cairns and Blake (2019).

A further problem with these projections is that the projection horizons are limited by the sample age range. For example, given that the maximum age in the sample age range is 100, one can only project out to a maximum age of 100. This problem implies that we cannot use mortality models with age effects *as they currently are* to value financial instruments whose values depend on the q rates of the extremely old. In the present case, we could use the model to price term annuities whose maximum term did not extend beyond age 100, but we could not use the model to price lifetime annuities or equity release mortgages. We would suggest that this limitation is a significant one, but can easily be rectified by using an age projection approach of the sort described above.

This article shows how q rate projections can be both improved to produce better behaved projection curves and extended to any future age, including ages well beyond the maximum age in the sample range.[2]

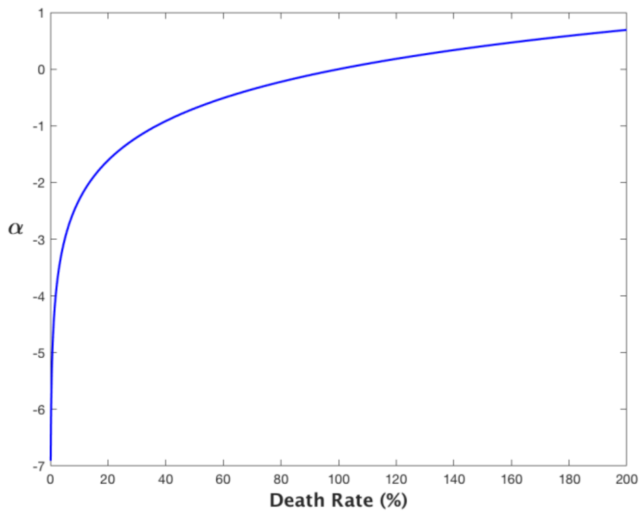
The article is organised as follows. Section 2 revisits some basic theory, section 3 provides an empirical example and section 4 concludes.

2. Theory: Age State Variable, Death Rate and Mortality Rate

Define the death rate $m(t,x) = D(t,x)/E(t,x)$, where $D(t,x)$ is a matrix of the number of deaths of individuals aged x in year t, and is the corresponding exposures matrix, i.e., $E(t,x)$ is a matrix of the number of individuals aged x in year t.

Define $\alpha(t,x) = logm(t,x)$ as the age SV for age x and year t. Figure 2 shows a standard log plot of $\alpha(t,x)$ vs the death rate $m(t,x)$.

Figure 2: α vs Death Rate



[2] An earlier application of this proposed age SV projection method to term annuity pricing is provided by Dowd *et al*., (2019b).

For obvious reasons, the theoretical death rate must always be bounded above by 100%. However, the empirical death rate can exceed 100% because of the possibility of measurement errors in the exposures data (see, e.g., Cairns *et al*., 2016). Accordingly, the Figure allows for possible death rates in excess of 100% on the x-axis. For convenience, we now drop the "(t, x)" terms when they are clearly redundant. Note also that α turns positive when m exceeds 100%. Thus, we should regard m >100% or equivalently α >0 as empirical possibilities that are associated with flawed data for extreme old age.

Since we are interested in death rates varying from 0 to 100% or a little more, the Figure establishes that we should be interested in the α range from -7 to somewhere a little above 0, say 1 or 2.

Having established this α range of interest, Figure 3 shows the transposed plot of the death rate vs α. However, we are not so much interested in the death rate m as in the mortality rate q~~, where $q = 1 - e^{m}$~~. Figure 4 shows the corresponding plot of the q rate vs the m rate. Observe the concave relationship between the two rates. Even at an m rate of 200%, which is empirically 'off the scale', the q rate is still short of 90%. Indeed, one would have to have m rates getting close to 500% to get q rates that approach 100%, as Figure 5 shows. Figure 6 reports the corresponding plot of the q rate vs α.

Figure 3: Death Rate vs α

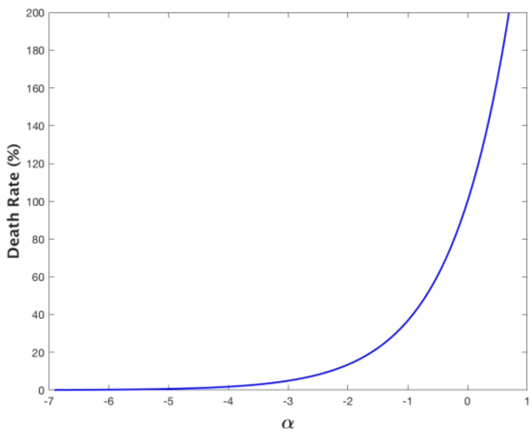


Figure 4: Mortality Rate vs Death Rate

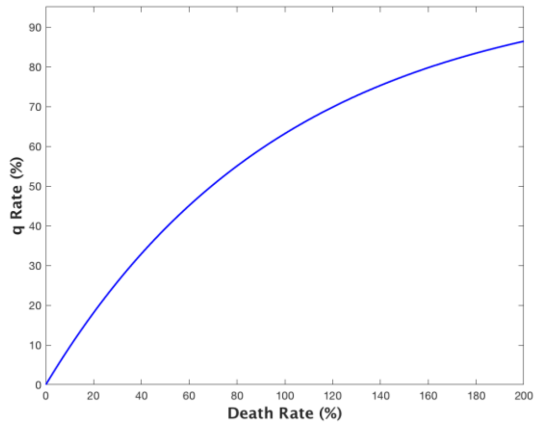


Figure 5: Mortality Rate vs Death Rate II

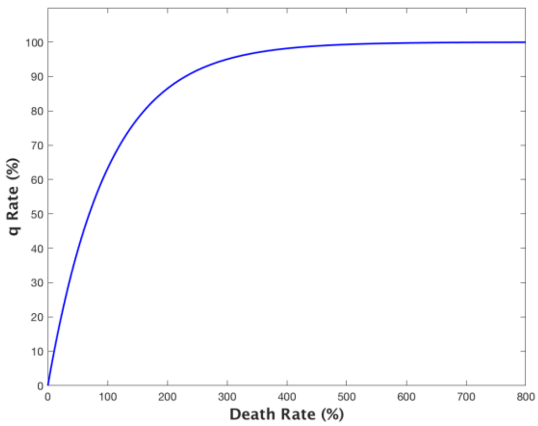
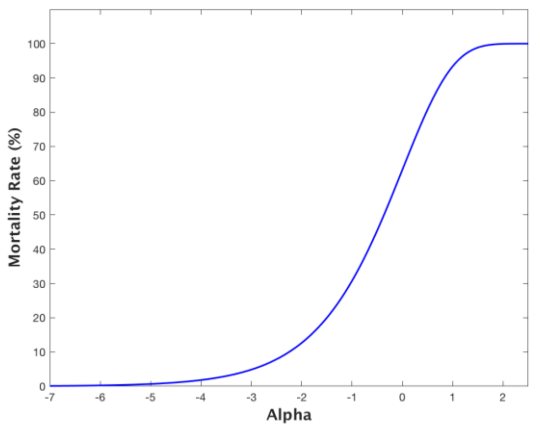


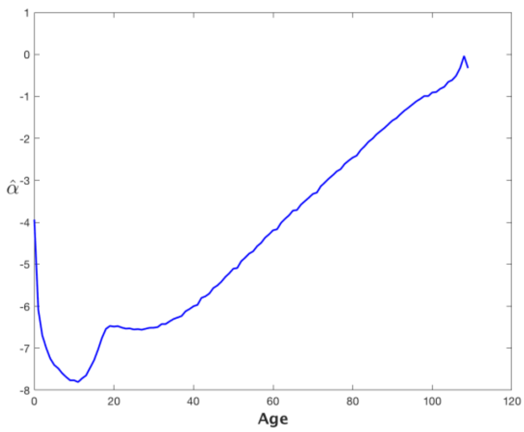
Figure 6: Mortality Rate vs Alpha



3. Projecting Extreme Old Age Mortality Rates: An Empirical Example

Figure 7 shows the familiar plot of the estimated α for Australian males for ages varying from 0 to 109. $\hat{\alpha}$ falls sharply after childbirth before turning upwards in the teenage years, levelling off in the late teens and then declining again in the early twenties (the 'accident hump'); it starts to rise again in the late 20s and continues rising thereafter. Of particular interest is the way in which $\hat{\alpha}$ becomes more volatile from the late 90s onwards – notice especially the hook-shaped tail – reflecting the estimates' increasing sensitivity to sampling variation as the age continues to increase.

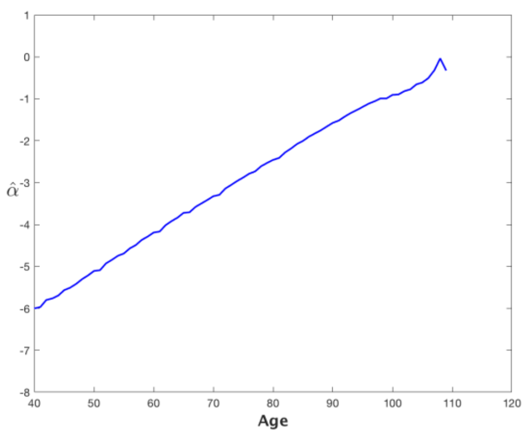
Figure 7: Australian Male $\hat{\alpha}$: Ages 0 to 109



Notes: Based on the CBDX3 model applied to Australian male deaths and exposures data for sample years 1921:2014 and sample ages 40:109. Source: Human Mortality Database.

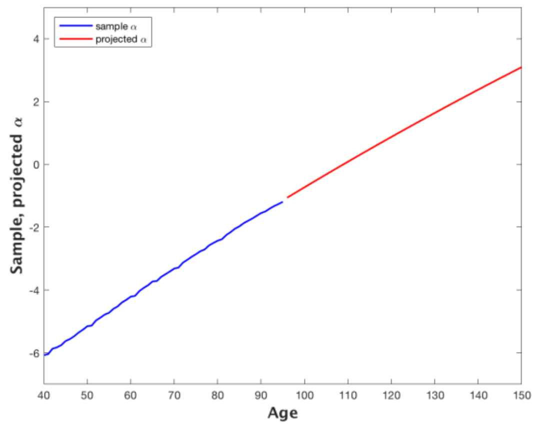
Figure 8 shows the same plot of estimated α for Australian males for ages varying from 40 to 109. Note the near linearity of the plot up to ages in the late 90s. This near-linear fit provides the basis for the α projections to higher ages shown in Figure 9. This Figure shows a blue plot of the sample $\hat{\alpha}$ going from ages 40 to 95. The red plot depicts the α projections going out to age 150. This second plot is a polynomial projection from the sample $\hat{\alpha}$ and we see that the projection is a well-fitting continuation of the sample $\hat{\alpha}$. Observe too that the projection is smooth and free of the random variation in the sample $\hat{\alpha}$.

Figure 8: Australian Male $\hat{\alpha}$: Ages 40 to 109



Notes: Based on the CBDX3 model applied to Australian male deaths and exposures data for sample years 1921:2014 and sample ages 40:109. Source: Human Mortality Database.

Figure 9: Australian Male $\hat{\alpha}$ and Projected α: Ages 40 to 109



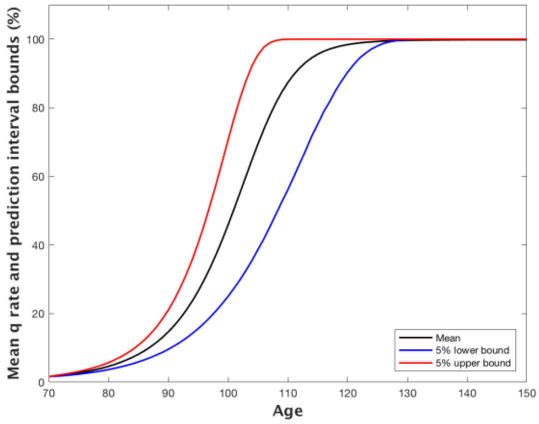
Notes: See notes to Figure 8.

We now propose the following method to obtain projected q rates going out to age 150. First recall that the projected q rates in Figure 1 were based on sample $\hat{\alpha}$. Our approach is to replace the sample $\hat{\alpha}$ (which here would be those for ages 40 to 95) with the polynomial fitted α underlying Figure 9, and we found that a quadratic equation gave the best fit.[3] We then use the fitted α to project the α for the ages higher than 95, and these are shown as the red line in Figure 9. Finally, we splice the fitted and projected α series to produce an α series spanning ages 40 to 150 and we input this spliced α series into (1) to obtain our projected q rates.

The resulting projections for the mean and 90% confidence interval for cohort q rates are shown in Figure 10. The q projections and their bounds rise with age and eventually converge to 100% as the age continues to rise. The projections are smoother and more intuitively appealing than those in Figure 1.

Figure 10: Projected Mean and 90% Prediction Intervals for Cohort q Rates for an Australian Male Just Turned 70

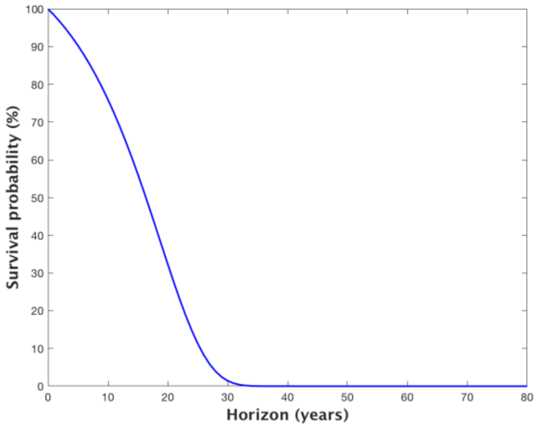
[3] The equation is $\alpha^{fitted}(x) = -0.0001x^2 + 0.1069x + 10.202$.



Notes: Based on the CBDX3 model applied to Australian male deaths and exposures data for sample years 1921:2014 and sample ages 40:100. Source: Human Mortality Database. Projections make use of a spliced α series spanning years 70 to 150 that includes fitted α for ages 70:95 and projected α for ages 96:150.

Figure 11 shows the projected survivorship probabilities corresponding to the q projections in Figure 10 for an individual just turned 70.

Figure 11: Survival Probabilities for Australian Males Just Turned Age 70



Notes: See notes to Figure 10.

Table 1 shows the survival probabilities to key benchmark ages: 80, 90, 100, etc.

Table 1: Survival Probabilities for Australian Males Just Turned Aged 70

Probability of survival to age 80	76.0%
Probability of survival to age 90	32.3%
Probability of survival to age 100	1.4%
Probability of survival to age110	1.20e-05%
Probability of survival to age 120	1.00e-18%
Probability of survival to age 130	6.30e-40%
Probability of survival to age 140	1.17e-65%

Probability of survival to age 150	5.84e-93%

Notes: See notes to Figure 10. Survival probabilities are based on mean *q* projections.

So the probability of surviving to age 100 is just over 1.4% and the probability of surviving to age 150 is about 5.84% with the decimal point moved 93 places to the left. To put this latter figure into perspective, the probability of surviving to age 150 is about 1/2000th of the probability of winning the national lottery 14 times in a row – possible but not too likely.

4. Conclusions

This article shows how the projected cohort mortality rates from stochastic mortality models that depend on age effects (or age state variables) can be improved by fitting and projecting the age effects themselves. The proposed approach produces smoother projected mortality rates and allows modellers to project cohort mortality rates out to ages well beyond the sample age range. This same approach can also be used to price financial instruments that depend on projected cohort mortality rates that eventually decline to zero, and the most obvious example would be to price a lifetime annuity. The proposed approach is thus of considerable practical use to mortality modellers, life insurers and pensions economists.

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