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Pension Schemes as Options on Pension Fund Assets: Implications for Pension Fund Management

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Pension schemes as options on pension fund assets: implications for pension fund management

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Abstract

This paper shows that the three principal types of funded pension scheme (defined benefit, defined contribution and targeted money purchase) are related through a set of options on the underlying financial assets held in the fund. The value of these options depends on both contribution inflows and the financial asset allocation chosen by the fund manager. The option values can therefore be used to assess both the appropriateness of the funding level and the effectiveness of the asset allocation in achieving the objectives of asset–liability management. In particular, they can be used to determine the probability of scheme insolvency. ©1998 Elsevier Science B.V. All rights reserved.

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1. Introduction

A funded pension scheme is composed of a pension fund *plus* a pension annuity. What differentiates one type of pension scheme from another is the set of rules governing the calculation of benefits when the scheme member retires. The simplest type of scheme is the *defined contribution (DC) scheme*: this uses the full value of the fund's assets to determine the amount of pension which, depending on the success of the fund manager, might be high or low. The *defined benefit (DB) scheme*, in contrast, calculates the benefit in relation to factors such as final salary, length of pensionable service and age of member, rather than to the value of the assets in the fund. For example, a typical UK scheme provides a pension equal to one-sixtieth (1.67%) of final salary for each year of pensionable service up to a maximum of 40 years' service; thus the maximum pension is two-thirds of final salary. In the case of a *targeted money purchase (TMP) scheme*, the aim is to use a defined contribution scheme to target a particular pension at retirement (which may be the same as that resulting from a final salary scheme), but which also benefits from any upside potential in the value of the fund assets above that required to deliver this target level. In other words, the TMP scheme aims to provide a minimum pension but not a maximum pension.

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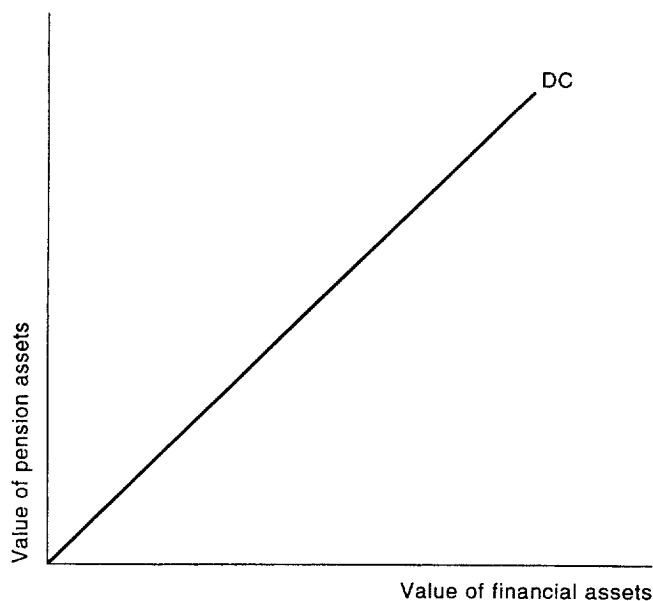


Fig. 1. A defined contribution pension scheme.

In Section 2, we examine the relationship between the three schemes in terms of the differing sets of options implicit in their structure,¹ while Section 3 shows how these options are valued. In Section 4, we investigate the rewards and risks faced by members, sponsors and fund managers from their participation in the different types of scheme. The option composition of the pension schemes and the reward–risk preferences of the pension scheme participants both provide a guide to the most suitable form of pension fund management as discussed in Section 5. Properties of the options can be used to measure the success of this fund management strategy, in particular, by providing an estimate of the probability of scheme insolvency for any given level of funding or asset allocation. In Section 6, we illustrate the strategy using hypothetical data, and we draw conclusions in Section 7.

2. The option composition of pension schemes

The differences between the schemes on the retirement date of the member are shown in Figs. 1–3. Fig. 1 shows that the present value of the DC pension on the retirement date depends entirely on the value of the fund assets on that date. Fig. 2 shows that the present value of the DB pension (L) is independent of the value of the fund assets, while Fig. 3 shows that the TMP pension has a minimum present value of L , but is higher if asset values exceed L .

Fig. 4 shows that the DB pension can be replicated using a long put option (P) and a short call option ($-C$) on the underlying assets of the fund (A), both with the same exercise price (L). The put option is held by the scheme member and written by the scheme sponsor, while the call option is written by the member and held by the sponsor. On the retirement date of the member, which coincides with the expiry date of the options, one of the options is (almost) certain to be ‘exercised’. If the value of the assets is less than the exercise price, so that the scheme is showing an actuarial deficit, the member will exercise his or her put option against the sponsor who will then be required to make a deficiency payment ($L - A$). If, on the other hand, the value of the assets exceeds the exercise price, so that the scheme is showing an actuarial surplus, the sponsor will exercise his or her call option against the member and recover the surplus ($A - L$). A member of a DB scheme therefore bears no asset market risk.

¹ The original treatment of pension fund liabilities as options is contained in Bagehot (1972), Sharpe (1976) and Treynor et al. (1976).

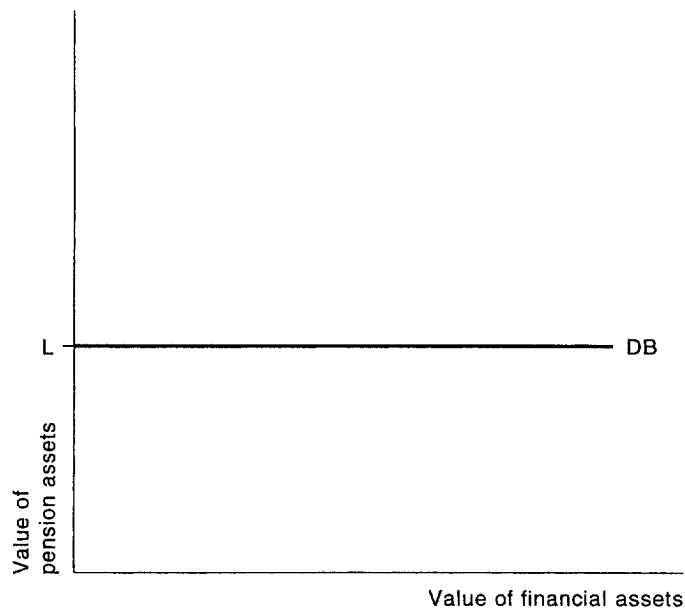


Fig. 2. A defined benefit pension scheme.

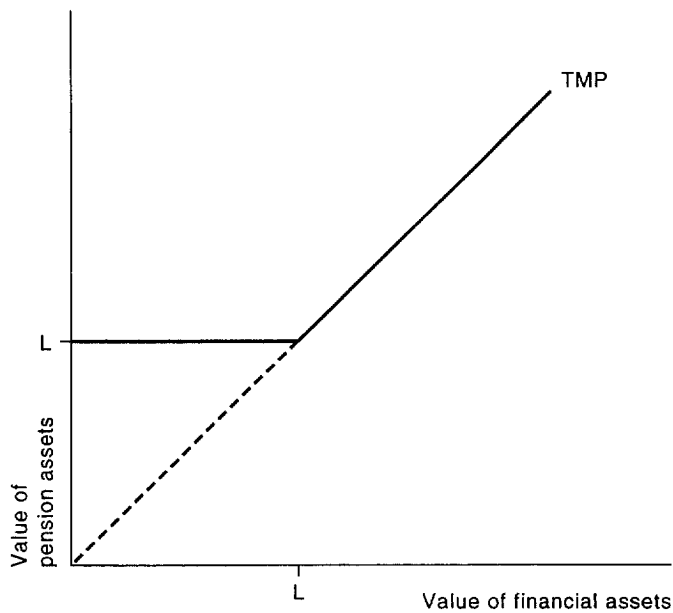


Fig. 3. A targeted money purchase pension scheme.

It is clear from this how DB and DC schemes are related. A DC scheme is invested only in the underlying assets. A DB scheme is invested in a portfolio containing the underlying assets (and so is, in part, a DC scheme) *plus* a put option *minus* a call option on these assets:

$$DB = L = A + P - C = DC + P - C. \quad (1)$$

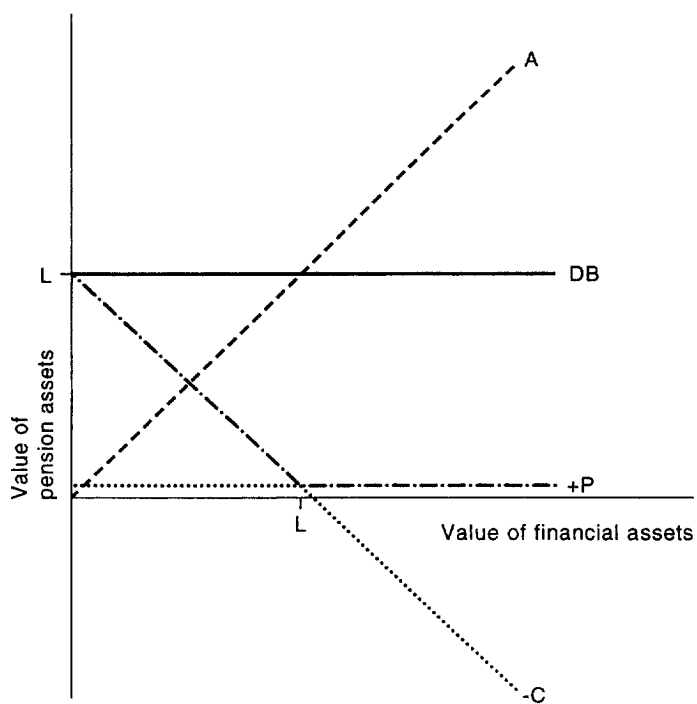


Fig. 4. The option composition of a defined benefit scheme.

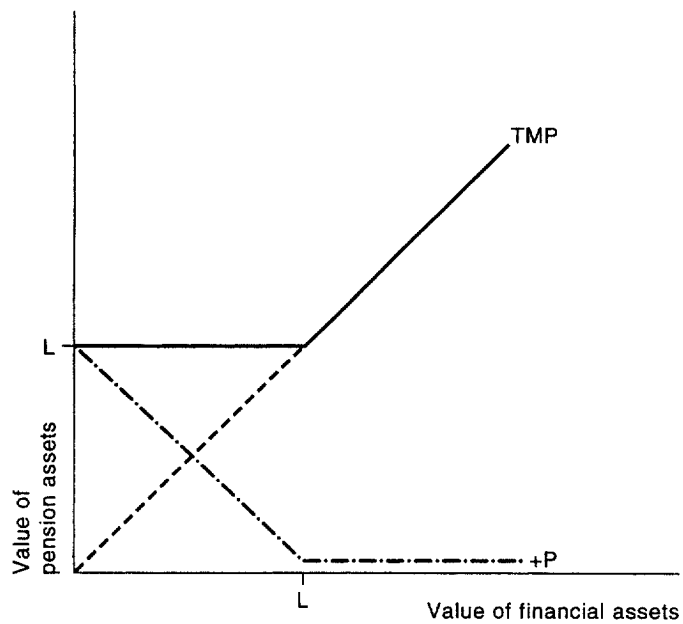


Fig. 5. The option composition of a targeted money purchase pension scheme.

Fig. 5 shows that the TMP pension can be replicated using a long (protective) put option (P) on the underlying assets of the fund (A) with an exercise price (L). The put option is held by the scheme member and written by the scheme sponsor. On the retirement date of the member, which again coincides with the expiry date of the option, the option will be exercised if the value of the assets is less than the exercise price. The effect of the option is to place a floor on the value of the pension received by the member. The present value of the TMP pension on the retirement date is the larger of the two present values provided by the DC and DB schemes, whatever the value of the underlying assets:

$$\text{TMP} = A + P = \text{MAX}(A, L) = \text{MAX}(\text{DC}, \text{DB}) = C + L. \quad (2)$$

This implies that a TMP scheme is equivalent to a call option (or floor) held by the member on the underlying pension fund assets with an exercise price L plus a riskless pure discount bond with a maturity value of L .² The call option will only be exercised if, on maturity, A exceeds L .

3. Valuing the options

We can now examine in more detail the structure of the pension schemes, and, in particular, show how the options are valued. We will concentrate on a DB scheme and assume throughout that the conditioning date is the start-up date of the scheme ($t = 0$), so that all values dated ahead of $t = 0$ will be expected values conditioned on information available at $t = 0$.

We will assume that the expected value of a scheme member's pension assets at any date t will equal the expected value of the accumulated financial assets (F_t) plus the expected discounted value of the remaining contributions until the retirement date (X_t).³ These, in turn, will depend on the member's starting income (Y_0), the contribution rate as a proportion of income into the scheme (γ), the expected future growth rate in income (g_Y) (which for simplicity we assume to be constant for the whole period), the expected yields on the investments in financial assets purchased with the contributions (r_{Fj}), the rate of tax relief on contributions (τ),⁴ the number of years of pensionable service (T) and the one-year survival probabilities from date $t = 0$ (p_t). Assuming that the appropriate discount rates used to discount the remaining contributions are the expected returns on financial assets held during the relevant period (which will depend on the fund management strategy pursued by the scheme as described in Section 5), the expected value of a member's pension assets at any date t is given by:⁵

$$A_t = F_t + X_t = \sum_{k=1}^t \frac{p_k \gamma Y_0 (1 + g_Y)^{k-1}}{1 - \tau} \prod_{j=k+1}^t (1 + r_{Fj}) + \sum_{k=t+1}^T \frac{p_k \gamma Y_0 (1 + g_Y)^{k-1}}{(1 - \tau) \prod_{j=t+1}^k (1 + r_{Fj})}, \quad t = 1, T, \quad (3)$$

where the symbol \prod represents the product of the terms immediately to the right (except for terms associated with $j > t$ which are set to unity since we assume that all cash flows arise at the end of the relevant period).

² We assume here that the target pension with the TMP scheme is the same as that with the DB scheme, but this need not be the case in general. It is also clear that the TMP scheme is equivalent to an endowment insurance scheme (see, e.g., Gemmill, 1993, Section 10.3). In the US, it is known as a floor-offset scheme.

³ This paper uses the *implicit lifetime contract method* or *prospective benefits funding method* of determining pension liabilities and assets (see, e.g., Disney and Whitehouse, 1996, or Haberman and Sung, 1994). This method assumes that the member will work until normal retirement age and then draw a pension until death. This contrasts with the *accrued benefits funding method* which determines pension liabilities and assets only up to the date of accrual and disregards likely future service (see, e.g., Institute and Faculty of Actuaries, 1984).

⁴ We assume for simplicity that the tax rates are the same for the member and the sponsor.

⁵ This is consistent with conventional practice by the economics profession (e.g., Tepper, 1981). However, other economists have used the after-tax rate of return on corporate bonds, e.g., Copeland (1984). This is more in line with the practice of the accountancy profession which recommends using the yield on long-term government bonds (see, e.g., Federal Accounting Standard 87 and International Accounting Standard 19).

With a DB scheme, the liabilities at retirement depend on the expected pension at retirement (Z), the expected growth rate in the pension (g_Z) and the one-year survival probabilities in retirement (p_{T+t}). Suppose that the retirement pension, Z , is equal to some proportion (θ , e.g., two-thirds) of the expected income at retirement ($Y_0(1 + g_Y)^{T-1}$). Therefore, the expected value of the liabilities at any date t is given by:⁶

$$L_t = \sum_{k=1}^{\infty} p_{T+k} Z \left[\frac{1 + g_Z}{1 + r_B} \right]^k \left[\frac{1}{\prod_{j=t+1}^T (1 + r_{Fj})} \right], \quad t = 1, T. \quad (4)$$

The expected actuarial surplus with a DB scheme is defined as the difference between (3) and (4):

$$S_t = A_t - L_t, \quad t = 1, T. \quad (5)$$

With a DC scheme, Eq. (3) equals the present value of both the assets and liabilities, so that there is no actuarial surplus. With a TMP scheme, the liabilities are the larger of Eqs. (3) and (4), but, as with a DC scheme, there is no surplus.

The options embodied in the DB and TMP schemes have the following characteristics. They are European options, since they cannot be exercised before the retirement date.⁷ In addition, the underlying asset does not make payouts prior to the expiry date of the option. However, the most important feature of the options is that the exercise price is not constant, as in the standard Black–Scholes model (Black and Scholes, 1973), but is equal to the value of the liabilities. The appropriate option valuation model is based on a modification to the Black–Scholes framework which recognises that the options in Eq. (1) or Eq. (2) are *exchange options*, i.e. options to exchange risky assets at an exercise price that is indexed to the uncertain value of the liabilities (see Fischer, 1978; Margrabe, 1978).

The value of the call option in Eq. (1) is given by:

$$C_t = N(d1_t)A_t - N(d2_t)L_t, \quad (6)$$

where:

$$d1_t = \frac{\ln(A_t/L_t) + 0.5\sigma_{S_t}^2(T-t)}{\sigma_{S_t}\sqrt{T-t}}, \quad (7)$$

$$d2_t = d1_t - \sigma_{S_t}\sqrt{T-t}, \quad (8)$$

$$\sigma_{S_t}^2 = \sigma_{A_t}^2 + \sigma_{L_t}^2 - 2\sigma_{ALt}, \quad t = 1, T. \quad (9)$$

$N(d1_t)$ and $N(d2_t)$ are normal distribution functions evaluated at $d1_t$ and $d2_t$, respectively. Eq. (9) is the variance ($\sigma_{S_t}^2$) of the surplus (5) (i.e. (square of) *surplus risk*) which depends on the standard deviations of the rates of return on assets (σ_{A_t}) and liabilities (σ_{L_t}), and the covariance between returns on the assets and liabilities (σ_{ALt}).⁸

We need to consider the most appropriate way of modelling the components of (9). This would appear to involve identifying sources of variability (hopefully small in number) common to both assets and liabilities. What follows is the simplest possible stylised framework for achieving this. Inevitably, substantial simplifying assumptions are involved. Where these assumptions do not correspond well with reality, a more realistic, but also a more complex, framework would be needed. We will suppose that the key sources of volatility facing both assets and liabilities are the volatilities attached to interest rates and growth rates, and that these volatilities will be scaled by the different

⁶ The discount rates from the retirement date onwards are the discount rates (r_B) on riskfree government bonds with a maturity of around 15 years (on the grounds that such bonds are used to finance annuities). The discount rates from the retirement date back to date t are the same as those used for discounting projected contributions, namely the expected returns on the financial assets in the pension fund. This accords with conventional actuarial practice.

⁷ More sophisticated versions of the model could contain options which allow for earlier termination of the pension scheme on the grounds of redundancy (exercised by the sponsor) or ill-health (exercised by the member), etc.

⁸ See also Leibowitz (1986b), who argues that liabilities can be treated as short positions in assets and liability returns can be treated in a commensurate way

durations of the assets and liabilities (see Macaulay, 1938). We also allow for there to be specific components to the asset and liability volatilities.

From Eq. (3), we can see that the volatility of the rate of change in the value of pension assets depends on their duration, which equals the weighted sum of the durations of the existing financial assets (D_{Ft}) and of the remaining contributions (D_{Xt}):⁹

$$D_{At} = \alpha_t D_{Ft} + (1 - \alpha_t) D_{Xt} = \alpha_t D_{Ft} + (1 - \alpha_t) \left[\sum_{k=t+1}^T \frac{(k-t)p_k \gamma Y_0 (1+g_Y)^{k-1}}{(1-\tau)X_t \prod_{j=t+1}^k (1+r_{Fj})} \right], \quad t = 1, T \quad (10)$$

(where $\alpha_t = F_t/(F_t + X_t)$ is the weight of the existing financial assets in total pension assets at time t), and the standard deviations of the rates of change in the yields on financial assets (σ_r) and in the growth rate in earnings (σ_g). As a first-order approximation, the variance of pension asset returns is given by:

$$\sigma_{At}^2 = D_{At}^2 (\sigma_r^2 + \sigma_g^2) + \eta_A^2, \quad t = 1, T. \quad (11)$$

where η_A is the specific risk on pension asset returns.¹⁰ We assume for simplicity that: financial asset returns and growth rates are uncorrelated; the standard deviation of the rate of change in financial asset returns is constant over time; and the standard deviations of earnings growth (g_Y), pensions growth (g_Z), and later inflation (g_I) and dividend growth (g_E) are all constant over time and equal to each other (which implies that the four growth rates differ, if at all, by constant amounts). These are clearly very strong assumptions and are unlikely to hold exactly in the real world. Nevertheless, they are useful assumptions to make if we wish to derive a tractable model.

⁹ The duration of the financial assets is equal to the value-weighted sum of the durations of the individual assets in the portfolio, see Blake (1995), Chapter 13, (Eq. (11)). The duration measure is highly sensitive to the underlying model of the term structure of interest rates used, as shown by (Boyle, 1978). For example, models with parallel yield curve shifts result in long-term assets having substantially greater durations than models with mean reversion. Nevertheless, Reitano (1991) has shown that there is an 'equivalent parallel yield curve shift' corresponding to any underlying yield curve shift and this enables conventional Macaulay-type measures of duration to be used.

¹⁰ Eq. (11) is derived as follows. Assume that in (3) the r_{Fj} are expected to be constant over time and the p_k are fixed at p , and define $d = p\gamma Y_0/(1-\tau)$. For $t = 0$, A_0 in (3) can be written:

$$A_0 = X_0 = \sum_{k=1}^T \frac{d(1+g_Y)^k}{(1+r_F)^k}.$$

The elasticity of A_0 with respect to $(1+r_F)$ is given by

$$\frac{\delta A_0}{\delta(1+r_F)} \frac{(1+r_F)}{A_0} = -\frac{1}{A_0} \sum_{k=1}^T \frac{k d(1+g_Y)^k}{(1+r_F)^k} \equiv -D_{A_0}.$$

The elasticity of A_0 with respect to $(1+g_Y)$ is given by

$$\frac{\delta A_0}{\delta(1+g_Y)} \frac{(1+g_Y)}{A_0} = \frac{1}{A_0} \sum_{k=1}^T \frac{k d(1+g_Y)^k}{(1+r_F)^k} \equiv D_{A_0}.$$

The total differential of A_0 can be written:

$$\frac{dA_0}{A_0} = \frac{\delta A_0}{\delta(1+r_F)} \frac{(1+r_F)}{A_0} \frac{dr_F}{(1+r_F)} + \frac{\delta A_0}{\delta(1+g_Y)} \frac{(1+g_Y)}{A_0} \frac{dg_Y}{(1+g_Y)} + \epsilon_A = -D_{A_0} \left[\frac{dr_F}{(1+r_F)} - \frac{dg_Y}{(1+g_Y)} \right] + \epsilon_A,$$

where we include a serially and contemporaneously uncorrelated specific risk component to the rate of return on pension assets. The volatility of the rate of return on pension assets is given by

$$\sigma_{A_0}^2 = D_{A_0}^2 (\sigma_r^2 + \sigma_g^2) + \eta_A^2.$$

Note that σ_r^2 and σ_g^2 are the volatilities of the rates of change in interest rates and growth rates, rather than the volatilities of their levels so that these will take relatively low values. Similar derivations apply for σ_{At}^2 , $t > 0$, and for σ_{Lt}^2 and σ_{ALt} in (13) and (14) below.

In a similar way, the volatility of the rate of change in the value of the pension liabilities depends, from Eq. (4) on their duration (see, e.g., Langtieg et al., 1986):

$$D_{Lt} = \sum_{k=1}^{\infty} \frac{kp_{T+k}Z}{L_T} \left(\frac{1+g_Z}{1+r_f} \right)^k + (T-t), \quad t = 1, T. \quad (12)$$

As a first-order approximation, the variance of the liability returns is given by

$$\sigma_{Lt}^2 = D_{Lt}^2(\sigma_r^2 + \sigma_g^2) + \eta_L^2, \quad t = 1, T, \quad (13)$$

where the standard deviation of the growth rate in the pension is σ_g and η_L is the specific risk on liability returns. The covariance between asset and liability returns is given by:

$$\sigma_{ALt} = D_{At}D_{Lt}(\sigma_r^2 + \sigma_g^2) + \eta_{AL}, \quad t = 1, T. \quad (14)$$

where η_{AL} is the covariance between the specific risks on asset and liability returns.

The value of the put options in Eqs. (1) and (2) is derived from put–call parity as (using Eqs. (5) and (6)):

$$P_t = C_t + L_t - A_t = C_t - S_t = (1 - N(d2_t))L_t - (1 - N(d1_t))A_t, \quad t = 1, T. \quad (15)$$

Two important features of Eqs. (6) and (15) are that the option values do not depend explicitly on the riskless rate of interest as in the standard Black–Scholes model, and that the appropriate definition of risk is not the risk, given by (11), attached to the *pension assets* (3), but the risk, given by (9), attached to the *pension surplus* (5). Both these features follow because the pension liabilities provide a natural hedge for the pension assets against both interest rate and growth rate risks.

The rationale for the first feature comes from the Black–Scholes innovation of constructing a riskless hedge portfolio. In order to do this, it is necessary to hedge against changes in both the value of the underlying assets and the exercise price. Changes in asset values are hedged by holding the assets. The cost of this hedge is equal to the rate of return on the assets. However, because the assets themselves are held in the portfolio, the return from the portfolio exactly offsets the cost of the hedge against changes in asset values. Therefore, the rate of return on assets does not appear in the option pricing formula. Because the hedge portfolio is riskless and generates the riskless rate of return, only the riskless rate of interest appears in the standard Black–Scholes formula. Changes in the exercise price are hedged by holding in the portfolio assets whose returns are perfectly correlated with changes in the exercise price, i.e. with changes in the value of the liabilities. This is achieved by holding, as part of the main portfolio, a portfolio of assets that exactly tracks any changes in the value of the liabilities. We will denote this portfolio the *liability immunising portfolio* (LIP) and we will assume that it is possible for us to construct such a portfolio. While the LIP will be a risky component of the total portfolio, it will be riskless relative to the liabilities that it is immunising and so it too will generate the riskless rate of return. Therefore, the rate of return on the hedge portfolio is 0, since the return on the liability hedge exactly offsets the return on the asset hedge.

The rationale for the second feature comes from the fact that pension asset and liability values respond to shocks in a similar way. In Eqs. (3) and (4), we assumed that the main sources of shocks are unexpected changes in yields and growth rates (in earnings or pensions, say). So, for example, an unexpected increase in yields reduces the present values of both assets and liabilities, while an unexpected increase in growth rates has the opposite effect. The volatilities of yields and growth rates are common to both assets and liabilities. The differential effects of these volatilities on the variances of asset and liability returns comes from the differing durations of assets and liabilities (see (10) and (12)), as seen in Eqs. (11), (13) and (14). Substituting these into Eq. (9), we get:

$$\sigma_{St}^2 = (D_{At} - D_{Lt})^2(\sigma_r^2 + \sigma_g^2) + \eta_A^2 + \eta_L^2 - 2\eta_{AL}, \quad t = 1, T. \quad (16)$$

This shows that the volatility of the surplus depends on the squared *duration gap* between assets and liabilities, the variances of the rate of change in yields and growth rates, and the relationship between the specific risks on assets and liabilities. If the financial asset portfolio is constructed to have returns that are perfectly correlated with changes

in the value of the liabilities, then $\eta_A^2 = \eta_L^2 = \eta_{AL}$, and the terms involving η in (16) vanish; such a portfolio would be a LIP. If, in addition, the duration of the assets is kept equal to that of the liabilities, then surplus risk can be eliminated altogether. It is the convenient structure of Eq. (16) that justifies the simplifying assumptions that we have made above. A more complex version of (16) would be needed if these assumptions were not valid, but it is unlikely that surplus risk could be eliminated completely in this more general framework.

4. The pension scheme preferences of members, sponsors and fund managers

We are now in a position to examine the preferences for the three types of pension scheme by members, sponsors and fund managers.

From the member's viewpoint, the schemes have different costs, different expected returns and different risks. Suppose that a typical member's reward–risk preferences can be represented by an isoelastic utility function with a constant relative risk aversion parameter β (see, e.g., Merton, 1969).¹¹ This suggests that the risk–reward indifference curves for the three types of scheme are given as follows:

$$U_{DB} = g_Y - \frac{1}{2}\beta\sigma_g^2 \quad (17)$$

for the DB scheme,

$$U_{DC} = r_{Ft} - \frac{1}{2}\beta[D_{At}^2(\sigma_r^2 + \sigma_g^2) + \eta_A^2] \quad (18)$$

for the DC scheme, and

$$U_{TMP} = r_{Ft} - (P_t/L_t(T-t)) - \frac{1}{2}\beta[(D_{At} - D_{Lt})^2(\sigma_r^2 + \sigma_g^2) + \eta_A^2 + \eta_L^2 - 2\eta_{AL}] \quad (19)$$

for the TMP scheme, using Eq. (2).

The DB scheme offers the lowest expected return equal to the anticipated growth rate in the member's earnings, with risk measured by the volatility of those earnings. The DC scheme offers the highest expected return equal to the expected return on financial assets (which for the dynamic efficiency of the economy as a whole must exceed g_Y), but also has the highest risk. The TMP scheme has a lower expected return than the DC scheme because of the cost of buying the protective put option, but as a consequence has lower risk.

The ranking of preferences depends on the degree of risk aversion as follows:

$$\begin{aligned} -\infty < \beta \leq \beta_{1t} &\Rightarrow U_{DC} > U_{TMP} > U_{DB}, \\ \beta_{1t} < \beta \leq \beta_{2t} &\Rightarrow U_{TMP} > U_{DC} > U_{DB}, \\ \beta_{2t} < \beta \leq \beta_{3t} &\Rightarrow U_{TMP} > U_{DB} > U_{DC}, \\ \beta_{3t} < \beta \leq \infty &\Rightarrow U_{DB} > U_{TMP} > U_{DC}, \end{aligned} \quad (20)$$

where:

$$\beta_{1t} = \frac{2P_t/L_t(T-t)}{(2D_{At}D_{Lt} - D_{Lt}^2)(\sigma_r^2 + \sigma_g^2) + 2\eta_{AL} - \eta_L^2}, \quad (21)$$

$$\beta_{2t} = \frac{2(r_{Ft} - g_Y)}{D_{At}^2(\sigma_r^2 + \sigma_g^2) + \eta_A^2 - \sigma_g^2}, \quad (22)$$

$$\beta_{3t} = \frac{2(r_{Ft} - (P_t/L_t(T-t)) - g_Y)}{(D_{At} - D_{Lt})^2(\sigma_r^2 + \sigma_g^2) + (\eta_A^2 + \eta_L^2 - 2\eta_{AL}) - \sigma_g^2}, \quad t = 1, T. \quad (23)$$

¹¹ While the original Black–Scholes model and its off-shoots such as the Fischer–Margrabe model being used here were derived under the assumption that investors are risk-neutral, it is possible to show that a risk-neutral valuation of the option is still valid if, as we are assuming, the average scheme member exhibits constant relative risk aversion (see, e.g., Rubinstein, 1976; Breeden and Litzenberger, 1978; Brennan, 1979).

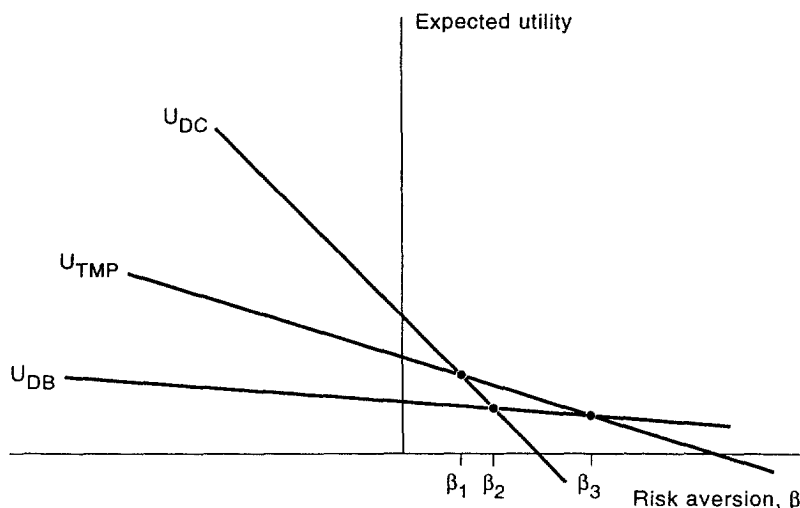


Fig. 6. Members' preferences over different pension schemes.

Both β_{2t} and β_{3t} will be positive and it is likely that β_{1t} is also positive, although if the duration of the assets is less than half that of the liabilities, β_{1t} will be negative. Individuals who are highly risk averse will prefer the DB scheme, those who are substantial risk takers will prefer the DC scheme, while those who are moderately risk averse and possibly even moderately risk taking will choose the TMP scheme. However, if the durations of assets and liabilities are continuously equalised, Eq. (23) shows that the TMP scheme will always be preferred to the DB scheme. This is demonstrated in Fig. 6.

Given these preferences by members for the different schemes, how are the risks shared between members, sponsors and fund managers? With a DC scheme, the position is straightforward: all the risk attached to the pension fund assets is borne directly by the member and none by the sponsor or fund manager, although in the long term the latter two will go out of business if they systematically deliver poor performance. With a DB scheme, the member bears no financial risk: he or she receives a pension that is based on some pre-set formula regardless of the value of the financial assets at retirement. All the downside risk is borne by the scheme sponsor; but the sponsor retains all the upside potential if asset performance is better than expected. The fortunes of the fund manager will be highly correlated with the extent to which deficits or surpluses are created. With a TMP scheme, all the downside risk is borne by the sponsor, while all the upside potential is retained by the member.

5. The optimal management of pension fund assets

We will again concentrate on a DB scheme and assume that the objective of the sponsors of such a scheme is to manage the pension assets over time so as to minimise the following function of surplus risk (16), subject to the constraints that the surplus (5) is 0 on the maturity date of the scheme and never falls below 0 prior to the maturity date:

$$\text{Min}_{\Omega_t} J_t = \sum_{k=t}^T \delta^{k+1-t} \sigma_{S^k}^2, \quad t = 1, T. \quad (24)$$

subject to:

$$S_t = A_t - L_t \geq 0, \quad t = 1, T-1, \quad (25)$$

$$S_T = A_T - L_T = 0. \quad (26)$$

In (24), δ is a discount factor lying between 0 and unity which allows for the possibility that surplus risk arising in the distant future might be of less concern to the scheme sponsor than surplus risk arising in the near term; we will, however, assume that the sponsor is equally concerned with surplus risk whenever it arises over the life of the scheme, so we set $\delta = 1$. Eq. (24) is minimised with respect to a set of control variables Ω_t which will be specified in more detail below.

Minimisation of surplus risk in a DB scheme is the rationale underlying *asset–liability management* (ALM), otherwise known as *surplus* (or *shortfall* or *solvency*) *risk management* (see, e.g., Leibowitz, 1986c, Leibowitz and Henriksson, 1987, Kritzman, 1988 and Bodie, 1991) and, for a recent overview, van der Meer and Smink, 1993). In contrast, the objective of the sponsors of a TMP scheme is to minimise the risk of generating a shortfall, while the sponsors of a DC scheme are not concerned either with the surplus or with surplus risk, but instead can choose the asset structure over time that maximises the member's utility function, given his or her degree of relative risk aversion (as in (18)).

There are two principal techniques for ALM: *immunisation* (Redington, 1952; Boyle, 1978) and *portfolio insurance* (Leland, 1980; Gatto et al., 1980; Brennan and Solanki, 1981; Leland and Rubinstein, 1981). The purpose of classical immunisation is to generate an assured return on the pension assets over the investment horizon. This is achieved by eliminating surplus risk, which, from Eq. (16), requires structuring the pension assets to have both the same duration as the liabilities and returns that are perfectly correlated with changes in the value of the liabilities, i.e. investing in a LIP. With classical portfolio insurance, in contrast, the sponsor seeks to lay off the downside risk that he or she faces from the exercise of the put that was sold to the member through the creation and management of a protective put option, while preserving the upside potential of the asset portfolio. However, the scheme member faces the opposite risk and may wish to protect his or her downside risk through the creation and management of a protective call option. We can therefore conceive of a portfolio insurance strategy (which we call *bi-directional portfolio insurance*) that attempts to eliminate the downside risks of both the sponsor and the member, but this will be at the cost of eliminating the upside potential in both cases. However, the fund management strategy needed to achieve bi-directional portfolio insurance will be identical to that required for classical immunisation as we now show.

Classical immunisation requires the surplus risk to be 0. However, when the surplus risk is 0, the values of the call and put are related solely to the size of the surplus (see Eqs. (6)–(9) and Eq. (15)). So if the pension scheme is being fully funded on a year-by-year basis and the asset portfolio is being continuously immunised to liabilities (so that the surplus is 0), the puts and calls both have a zero value since neither will be exercised. The pension scheme's fund manager can therefore replicate the payoff patterns of the put and call options (i.e. implement bi-directional portfolio insurance) by ensuring that the liabilities are immunised continuously over time using assets invested in a LIP.

However, as we shortly show, it will be impossible to invest in a LIP for the whole investment horizon, so the optimal asset allocation strategy will need to use a mixture of financial assets: equities, index bonds and, possibly, conventional fixed-interest bonds.

We will assume that the realised returns on these assets are determined by the following variation on the standard linear market model in finance:

$$r'_{Et} = r_f + \pi D_E + \epsilon_{Et}, \quad r'_{It} = [\rho_f(1 + g_I) + g_I] + \pi D_{It} + \epsilon_{It}, \quad r'_{Bt} = r_f + \pi D_{Bt} + \epsilon_{Bt}, \quad t = 1, T. \quad (27)$$

where π is the (stochastic) *duration risk premium*, D_E , D_{It} and D_{Bt} are the (deterministic) durations of equities, index bonds and conventional bonds, respectively, r_f is the (constant) nominal rate on treasury bills, ρ_f is the (constant) real rate on treasury bills and g_I is the expected inflation rate; expected returns are denoted r_E , r_{It} , and r_{Bt} . We further assume that the specific components of the returns on assets, ϵ_{Et} , ϵ_{It} and ϵ_{Bt} , have the following properties: they have zero mean, constant variance, and are serially and contemporaneously uncorrelated with each other, but that ϵ_{It} is perfectly correlated with the rate of change of the liabilities, so that $\eta_{It}^2 = \eta_L^2 = \eta_{IL}$. This is because index bonds (whose values are perfectly correlated with changes in the price level) are assumed to be perfectly correlated with pension liabilities (whose values are perfectly correlated with changes in the wage level),

because changes in wages and prices are, in turn, assumed to be perfectly correlated. Comparing (27) with the standard market model, π corresponds with the market risk premium (and, like the market risk premium, is assumed to be identical across securities) and duration corresponds with beta.

The motivation for this model again lies with the desire to utilise common sources of variability between assets and liabilities. In Section 3, we argued that the volatility of interest rates and growth rates would be key common sources of volatility. We will therefore assume that the variances of the returns on equities, index bonds and conventional bonds, and the covariances between them can be modelled as follows:

$$\begin{aligned}\sigma_E^2 &= D_E^2(\sigma_r^2 + \sigma_g^2) + \eta_E^2, & \sigma_{I_t}^2 &= D_{I_t}^2(\sigma_r^2 + \sigma_g^2) + \eta_{I_t}^2, & \sigma_{B_t}^2 &= D_{B_t}^2\sigma_r^2 + \eta_{B_t}^2, \\ \sigma_{EI_t} &= D_E D_{I_t}(\sigma_r^2 + \sigma_g^2), & \sigma_{EB_t} &= D_E D_{B_t}\sigma_r^2, & \sigma_{IB_t} &= D + D_{B_t}\sigma_r^2,\end{aligned}\quad (28)$$

where η_E , η_{I_t} , and η_{B_t} are the specific risk components of the returns on equities, index bonds, and conventional bonds. From (27) and (28), it is clear that the variance of π is given by $(\sigma_r^2 + \sigma_g^2)$: the variance of the market portfolio in the market model is replaced by the sum of the variances of its components in this model. Notice that the variance of fixed-income bond values depends only on the variance of interest rates.

We will assume for simplicity that the duration of equities is constant and given by

$$D_E = \frac{1 + r_E}{r_E - g_E}, \quad (29)$$

which holds under the assumption that g_E , the expected growth rate in dividends, is constant.¹² However, we will assume that index and conventional bonds with different durations are available for inclusion in the optimal portfolio.

Eqs. (24)–(26) constitute a dynamic programming problem, the solution to which we now outline. The first task is to find the equilibrium contribution rate over the life of the scheme. This is given by the value of γ that generates a zero surplus to the scheme on the member's retirement date (i.e. that satisfies (26)). So the first component of Ω_t is the contribution rate γ . However, it so happens that with a constant contribution rate satisfying (26), the scheme will show an actuarial surplus in each year prior to retirement, so that (25) will automatically be satisfied for all t . This result arises because of the *backloading* of benefits and hence contributions in DB schemes. In other words, a year of pension entitlement accruing late in the life of a scheme is more expensive to fund since its associated contribution has less time to benefit from compounded returns than a year of entitlement accruing early in the life of a scheme. For the surplus to be 0 in each period, the contribution rate would have to start off at a very low level, but would rise exponentially over the life of the scheme and end up well above the average for the whole period. If the contribution rate is held constant for the whole period at the level needed to fully fund the pension, then a surplus will build up early on, only to be run down to 0 as the member approaches retirement (for more details, see Blake and Orszag, 1997, Section 4.2 and Hemming, 1998, Appendix I).

The second task is to choose the financial asset allocation to minimise (24), conditional on the properties of the individual asset categories. Given its time-separable nature, (24) can be minimised on a year-by-year basis. We will show that the optimal portfolio of financial assets chosen by the fund manager will be determined in a sequence of up to *three* stages over the lifetime of the scheme, with the length of each stage depending on the relationship between the durations of the pension liabilities and pension assets (Samuelson (1989) has called this investment strategy *age-phasing* and it is also now known as *lifestyle* fund management).

To begin with, the duration of the pension liabilities will greatly exceed that of the pension assets (financial assets *plus* remaining contributions). This is because: the duration of the remaining contributions is lower than that of the liabilities, initially the weight of financial assets in total pension assets will be negligible, and also there do not exist financial assets with sufficiently high durations to compensate for this. It will therefore be impossible to invest

¹² See, e.g., Boquist et al. (1975). As calculated using this formula, the duration of equities will be substantially higher than that for bonds: it assumes that g_E does not respond to changes in r_E . Not all investigators believe that equity duration is as great as implied by this formula, e.g., Leibowitz (1986a).

entirely in a LIP from the start of the scheme. The objective at the first stage therefore is to build up the duration of the financial assets as quickly as possible. This involves investing the entire portfolio of financial assets in the highest duration assets of all, i.e. in equities.¹³ The second component of Ω_t is therefore D_E . The first stage is characterised by the following inequality holding:

$$D_{Lt} \geq D_{At} \equiv \alpha_t D_E + (1 - \alpha_t) D_{Xt}, \quad t = 1, T_1, \quad (30)$$

with (30) holding as an equality at T_1 which marks the end of the first stage.

The objective at the second stage is to build up the investment in the LIP, while preserving the equality between the durations of the pension assets and liabilities. The LIP will contain index bonds that are perfectly correlated with and have the same duration as the liabilities. It will generally not be possible at the second stage to switch immediately and fully into index bonds, since the duration of index bonds is likely to be less than that of equities; so again the investment in index bonds will have to be built up gradually in a way that preserves the equality of duration between assets and liabilities. The second stage is characterised by the following equality holding throughout:

$$D_{Lt} = D_{At} \equiv \alpha_t \psi_t D'_1 + \alpha_t (1 - \psi_t) D_E + (1 - \alpha_t) D_{Xt}, \quad t = T_1 + 1, T_2, \quad (31)$$

where

$$\psi_t = \frac{D_E - (D_{Lt} - (1 - \alpha_t) D_{Xt}) / \alpha_t}{D_E - D'_1} \quad (32)$$

is the weight of index bonds in total financial assets and where D'_1 is the maximum available duration on index bonds. Now ψ takes the value 0 at T_1 and the value unity at T_2 , at which point the duration of the liabilities equals the maximum available duration on the index bonds, and this marks the end of the second stage. So the third element of Ω_t is ψ_t .

Thus far we have accumulated financial assets in such a way that, by the end of the second stage, there is both duration-matching and perfect correlation between pension assets and liabilities. The first two stages involve investing in the riskiest assets in order to build up duration, regardless of the degree of risk aversion of the sponsor or member. This is counter-intuitive: we invest in the *riskiest* financial assets in order to *reduce the risk* to the pension liabilities. At the same time, we are taking advantage of the *equity risk premium* (πD_E).¹⁴ At the third stage, the objective is to continuously immunise liabilities using the LIP and simultaneously to achieve the desired reward-risk configuration on any *excess assets* above that needed for the LIP. Conditioning at time $t = 0$, excess assets will only be expected to arise if the sponsor, to guarantee a margin of safety, selects a higher contribution rate than the one needed to achieve a zero surplus at the retirement date. It is only under these circumstances that the fund manager is in a position to take into account the degree of risk aversion of the sponsor or the member, depending on the type of scheme. As we saw in Section 4, the lower the degree of risk aversion, the greater the degree of risk and hence expected return that can be sustained by the fund manager. With a DB scheme, it will be the sponsor's degree of risk aversion that is relevant, while with DC and TMP schemes, the optimal portfolio of financial assets will depend on the member's attitude to risk.¹⁵

Once the duration of the financial assets has been built up sufficiently to equal (taking into account the duration of the remaining contributions) the duration of the pension liabilities (so that surplus risk is 0), a highly risk averse

¹³ In practice, pension funds do not hold 100% of their assets in equities, but in the UK, they do hold around 80% in equities.

¹⁴ There are very few five-year periods in the history of any advanced financial system over which equities have not out-performed bonds. Further, Samuelson (1989, 1991, 1992) argues that, when security returns are mean-reverting, it is rational for long-horizon investors such as pension funds to invest more heavily in high-risk equities than in low-risk bonds during the early years of a pension scheme and then to switch into bonds as the horizon shortens. Indeed, it is arguable that over a long-investment horizon and with liabilities linked to growth rates in the economy, it is fixed-income bonds rather than equities that are the genuine high-risk asset. Only when security returns are pure random walks is it the case that the optimal asset allocation does not depend on the length of the investment horizon (see, e.g., Samuelson, 1963, and Merton and Samuelson, 1974).

¹⁵ Note that a DC scheme operates as a stage 3 scheme throughout its entire life.

scheme member or sponsor will be satisfied with the LIP as the only portfolio of financial assets needed. However, a less risk averse member or sponsor might wish to take on some additional *asset risk* by investing in a portfolio of risky assets, comprising equities, index bonds and conventional bonds; this is equivalent to taking on some additional diversifiable risk in an otherwise well-diversified portfolio.

The required weights in equities, index and conventional bonds in a portfolio if it is to be mean-variance efficient can be easily derived (see, e.g., Elton and Gruber, 1995, Chapter 6):

$$\begin{aligned}\lambda_{Et} &= \frac{D_E \eta_E^2 \eta_B^2}{D_E \eta_E^2 \eta_B^2 + D_{It} \eta_E^2 \eta_B^2 + D_{Bt} \eta_E^2 \eta_I^2}, & \lambda_{It} &= \frac{D_{It} \eta_E^2 \eta_B^2}{D_E \eta_E^2 \eta_B^2 + D_{It} \eta_E^2 \eta_B^2 + D_{Bt} \eta_E^2 \eta_I^2}, \\ \lambda_{Bt} &= \frac{D_{Bt} \eta_E^2 \eta_I^2}{D_E \eta_E^2 \eta_B^2 + D_{It} \eta_E^2 \eta_B^2 + D_{Bt} \eta_E^2 \eta_I^2}, & t &= T_2 + 1, T.\end{aligned}\quad (33)$$

The expected return on an efficient portfolio is given by

$$r_{Ft}^* = \lambda_{Et} r_E + \lambda_{Bt} r_{Bt} + \lambda_{It} r_{It}, \quad t = T_2 + 1, T. \quad (34)$$

and its standard deviation is given by

$$\sigma_{Ft}^* = \sqrt{[\lambda_{Et}^2 \sigma_E^2 + \lambda_{Bt}^2 \sigma_{Bt}^2 + \lambda_{It}^2 \sigma_{It}^2 + 2\lambda_{Et} \lambda_{Bt} \sigma_{EBt} + 2\lambda_{Et} \lambda_{It} \sigma_{EIt} + 2\lambda_{Bt} \lambda_{It} \sigma_{BIt}]}, \quad t = T_2 + 1, T. \quad (35)$$

The optimal portfolio is found at the point along the efficient frontier where the sponsor's (or the member's) marginal rate of substitution equals the price of risk:

$$\beta \sigma_{Ft}^* = \frac{r_{Ft}^* - r_f}{\sigma_{Ft}^*} \equiv \mu_t, \quad (36)$$

say, which implies that the expected return on the optimal portfolio can be determined using:

$$r_{Ft}^* = r_f + \mu_t \sigma_{Ft}^* = r_f + \mu_t^2 / \beta, \quad (37)$$

where μ_t^2 / β is the risk premium required by the member.

With D_E given, we have one degree of freedom in choosing either D_{It} or D_{Bt} . If the member or sponsor selects D_{It}^* , we can calculate the required duration on the conventional bonds in the optimal portfolio, using (37), (34) and (33), and the expected returns on assets derived from (27):

$$D_{Bt}^* = \text{Min} \left[D_B', \frac{(\mu_t^2 / \beta) \eta_E^2 \eta_I^2 + \sqrt{[(\mu_t^2 / \beta) \eta_E^2 \eta_I^2]^2 - 4\pi \eta_E^2 \eta_I^2 k_t}}{2\pi \eta_E^2 \eta_I^2} \right], \quad t = T_2 + 1, T, \quad (38)$$

where

$$k_t = (\pi D_E - (\mu_t^2 / \beta)) D_E \eta_E^2 \eta_I^2 + (\pi D_{It}^* - (\mu_t^2 / \beta)) D_{It}^* \eta_E^2 \eta_B^2 \quad (39)$$

and where D_B' is the maximum available duration on conventional bonds. Using this and (33) again, we can calculate the set of optimally changing weights in equities, conventional bonds and index bonds during the third stage between $T_2 + 1$ and T .

This concludes our derivation of the optimal ALM strategy. Before looking at an example that illustrates this strategy, we end this section with the following observations. The first concerns the size of any surplus in the scheme prior to retirement. The contribution rate that satisfies (26) above is the minimum possible contribution rate into the scheme to ensure its solvency at maturity. However, we have also shown that, due to the backloading effect, the scheme could be showing a surplus prior to maturity even if all expectations are fulfilled. So the existence of

a surplus prior to maturity is not necessarily a sign that a scheme is overfunded. Any surplus in a level-funded scheme prior to maturity should be calculated with reference to the ‘surplus’ from the minimum-funded scheme considered here. Yet regulatory authorities sometimes place what appear to be somewhat arbitrary restrictions on the permissible size of any surplus or deficit. For example, in the UK, any surplus exceeding 5% of liabilities must be eliminated over a 5-year control period via a reduction in the contribution rate (Social Security Act 1986). Any deficit exceeding 10% of liabilities must be reduced to below 10% within one year and any deficit up to 10% of liabilities must be eliminated within five years via an increase in the contribution rate (Pensions Act 1995). Such regulations may not be consistent with an ALM strategy that permits level funding over the life of a scheme.

The second point concerns the fulfilment of expectations. It is highly unlikely that expectations will be met in full because of the uncertainty attached to growth rates and asset returns, etc. Disastrous investment performance, for example, could lead to scheme insolvency. Fortunately, the properties of the options discussed above provide a measure of the probability of insolvency at any date. It is well known (see, e.g., Gemmill, 1993, Section 5.3) that $1 - N(d_{2t})$ in (15) gives the probability that the put option ends up in-the-money, in other words gives the probability of a deficit on the maturity date of the scheme. Further, the conditional expectation of the size of the deficit at maturity ($D_T = L_T - A_T$) is equal to the conditional expectation of the value of the put at maturity:

$$E_t(D_t) \equiv E_t(P_T) = L_t - \left(\frac{1 - N(d_{1t})}{1 - N(d_{2t})} \right) A_t, \quad t = 1, T. \quad (40)$$

The unconditional expectation of the size of the deficit is given by the product of Eq. (40) and $1 - N(d_{2t})$, but this just equals the current value of the put given in Eq. (15).

The scheme sponsor might be concerned that, with the current contribution rate, the size of the insolvency probability is too high and so chooses a higher contribution rate to reduce this probability. Also, with the passage of time, realised values for growth rates in earnings and discount rates, etc. will replace expected values and a realised surplus different from that which was expected will emerge. For the purpose of maintaining the solvency of the scheme, the contribution rate might have to be adjusted from time to time in order to amortise an unplanned surplus or deficit over a specified control period. However, scheme sponsors prefer to have a stable contribution rate over time to one that is highly volatile (see Lee, 1986). O’Brien (1986, 1987) and Haberman and Sung (1994) use a dynamic programming framework similar to (24)–(26) to derive the optimal time path of contributions into a DB scheme which minimises both contribution rate risk and solvency risk.

Finally, the above analysis has been conducted for a single-member scheme, but multi-member schemes are easy to handle since they involve a straightforward case of aggregation across members, taking into account their different stages of membership. We show in the next section that there are diversification benefits with multi-member schemes that help reduce the chance of insolvency.

6. Example

In this section, we will illustrate the optimal asset allocation strategy using a realistic example based on typical data for a DB scheme in the UK (see, e.g., BZW Equity-Gilt Study, 1997; Pension fund indicators, 1997; Blake, 1996; Neill, 1977).

We will assume that a male worker joins a pension scheme at age 25 years on the following terms and has survival probabilities based on English Life Tables 15:

- starting salary (Y)=£10 000 p.a.
- projected growth rate in salary (g_Y)=5.45% p.a.
- years to retirement (T)=40
- tax rate (τ)=25%
- pension fraction at retirement (θ)=66.67%
- projected pension at retirement (Z)=£ 54,924 p.a.

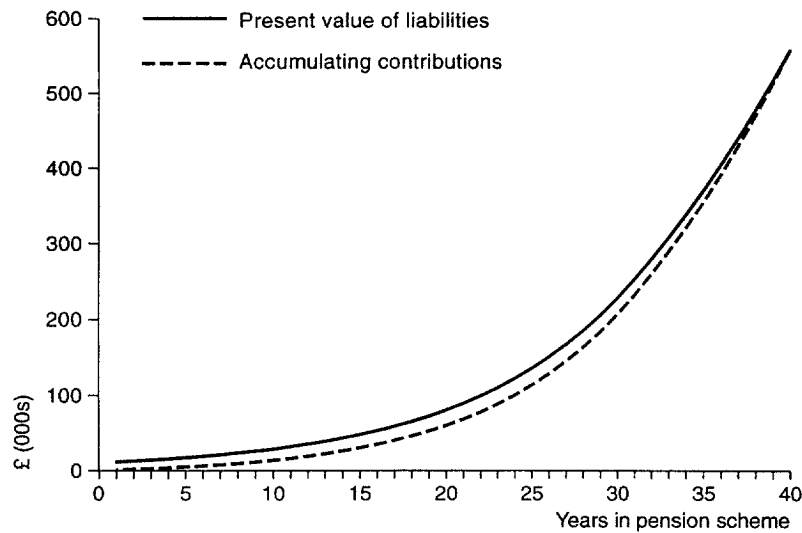


Fig. 7. Pension liabilities and contributions.

projected growth rate in pension (g_Z)=4% p.a.

projected inflation rate (g_I)=4% p.a.

degree of risk aversion (β)=1.25.

Financial assets with the following properties are available:

riskfree nominal interest rate (r_f)=5% p.a.

riskfree real interest rate (ρ_f)=1% p.a.

duration of equities (D_E)=20 years

projected growth rate in equity returns (g_E)=5.45% p.a.

maximum available duration of index bonds (D'_I)=10 years

maximum available duration of conventional bonds (D'_B)=8 years

standard deviation of rate of change in yields (σ_r)=0.75% p.a.

standard deviation of rate of change in growth rates, etc. (σ_g)=0.75% p.a.

duration risk premium (π)=0.3% p.a.

specific risk on equities (η_E)=10% p.a.

specific risk on index bonds (η_I)=5% p.a.

specific risk on conventional bonds (η_B)=4% p.a.

market risk premium (μ)=0.25 per percentage point of standard deviation.

Using these data, we can equate Eqs. (3) and (4) for $t = T$ and solve for the minimum required contribution rate (γ) at 4.86% of salary.

Fig. 7 plots the present value of the liabilities and the value of the accumulated contributions during each of the 40 years' membership of the scheme. By the 40th year, they are both valued at £ 557 729, sufficient to provide a pension of £ 54 924 per year (indexed to inflation) for the remainder of the member's life. Fig. 8 plots the present value of the remaining pension contributions for each year's membership of the scheme. This begins at £ 29 726 and declines initially slowly and then very rapidly over the remaining life of the scheme.

Fig. 9 plots the durations of the pension liabilities and remaining pension contributions for each year of the scheme. The duration gap between liabilities and remaining contributions declines from 15.8 years at the start of the scheme to 9.6 years at the end. This gap has to be filled as rapidly as possible with financial assets, but as Fig. 10 shows, the duration of financial assets required to do this is initially substantially greater than the maximum available duration, namely 20 years on equities. Fig. 11 shows how the duration gap is filled. It takes 31 years to

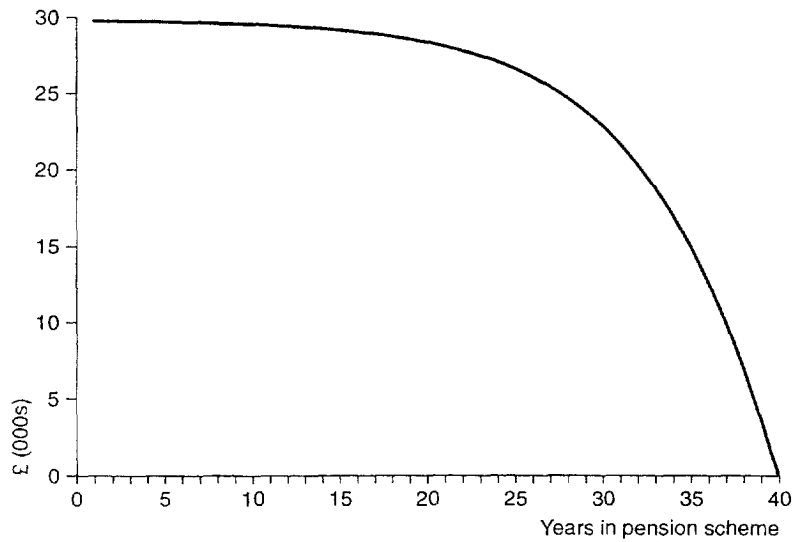


Fig. 8. Present value of remaining pension contributions.

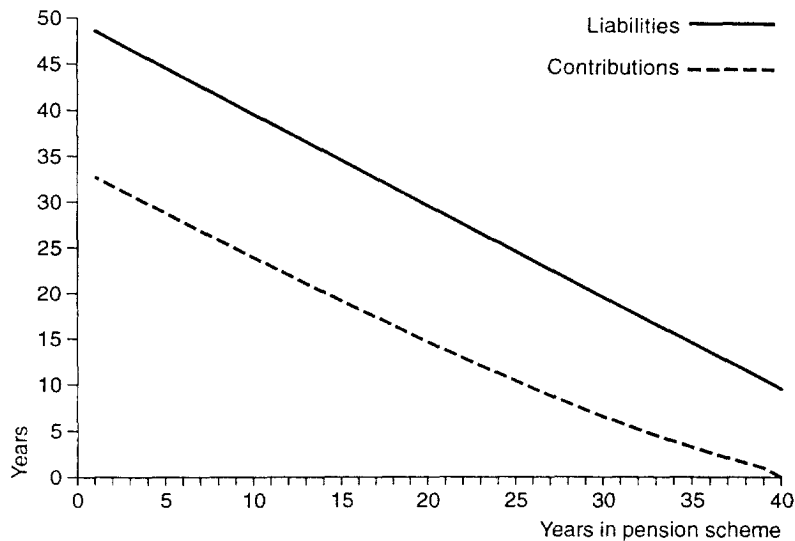


Fig. 9. Duration of pension liabilities and contributions.

build up sufficient financial assets with sufficient duration to eliminate the gap. This is stage 1 of the ALM strategy. The objectives of stages 2 and 3 are to build up the LIP and then to meet the reward–risk target of the member or sponsor, while preserving the duration gap at zero. However, if, as in this case, the scheme is funded on a minimum contribution only basis and realised investment returns do not exceed projected returns, then excess financial assets will not build up and there will be no third stage to the strategy.

Fig. 12 shows the changing financial asset composition that achieves these objectives over the life of the scheme. For the 31 years of stage 1, only equities are held in the portfolio of financial assets. The remaining nine years constitute stage 2 and it takes eight of these years to build up fully the investment in the LIP: the final year has 100% weighting in the LIP. Fig. 13 shows how these portfolio weights translate into portfolio expected returns and risks (as measured by standard deviations) over the life of the scheme. The two stages of the ALM strategy

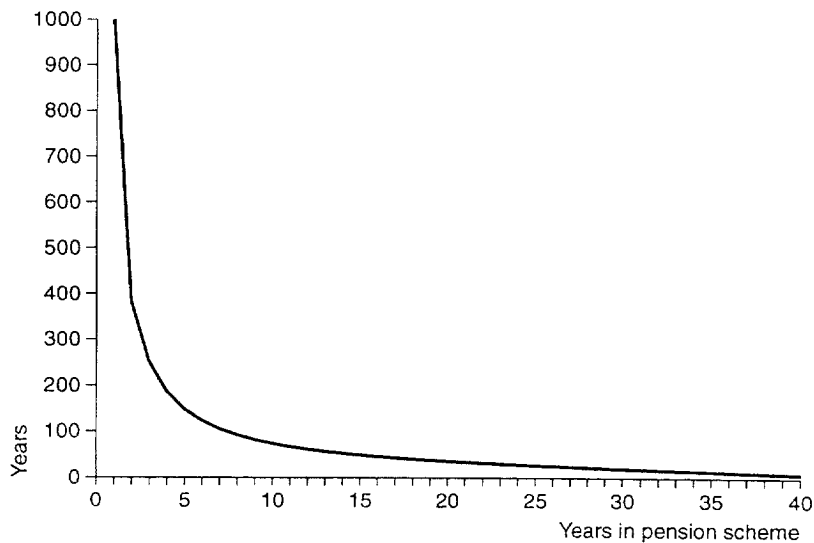


Fig. 10. Required duration of financial assets.

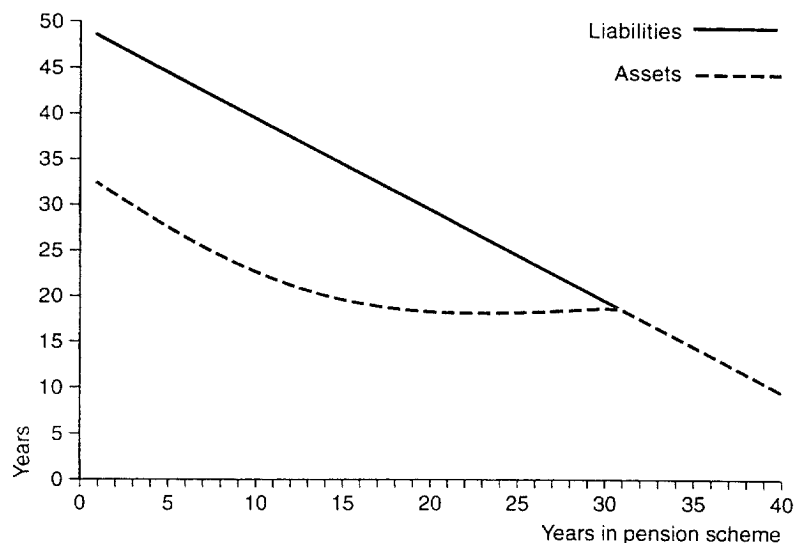


Fig. 11. Duration of pension liabilities and assets.

are clearly discernible, with the effects of increasing diversification leading to risk falling substantially more than returns during the final nine years.

Fig. 14 shows what happens to the pension fund surplus and to the values of the call and put during the life of the scheme assuming that expectations are fulfilled. With a level contribution rate of 4.86%, the surplus takes a value of £ 19 320 at the beginning of the scheme and this value declines monotonically over the life of the scheme to reach a zero surplus just prior to the member's retirement. This demonstrates clearly the effect of backloading in DB schemes. Even if a scheme is not overfunded and is just exactly funded over its lifetime, it will exhibit an apparent 'surplus' until very near the maturity date. From (15), it is clear that the surplus is equal to the difference between the call and put values and the figure shows the relationship between the three values over time. Starting at £ 21 850, the call rises in value for the 31 years of stage 1 of the ALM, reaching a peak of £ 33 440, before falling

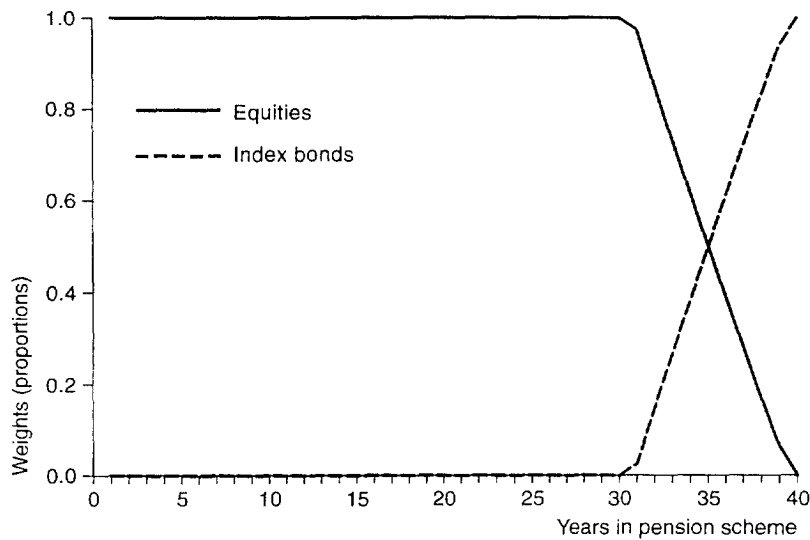


Fig. 12. Financial asset composition with minimum contributions.

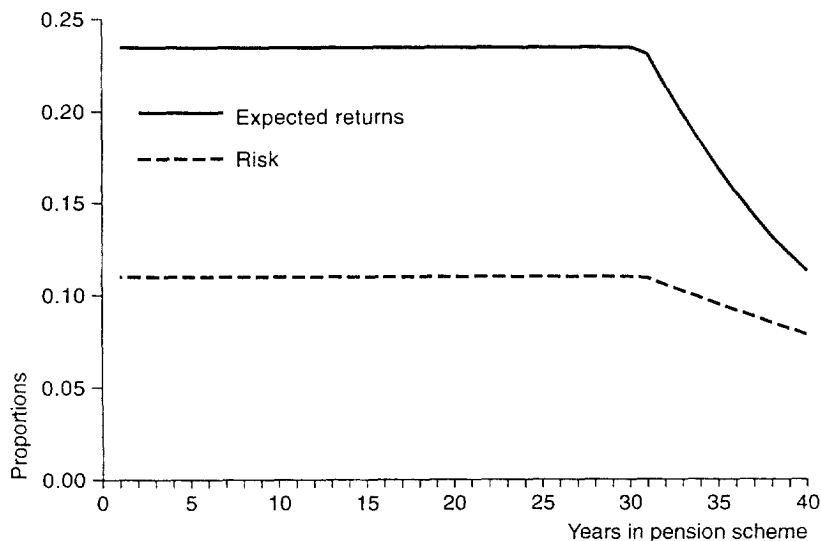


Fig. 13. Portfolio expected returns and risk with minimum contributions.

rapidly in value during the nine years of stage 2 as the backloading effect comes into play. The put, in comparison, is initially worth very little (£ 2 530), but its rate of increase in value during stage 1 is much greater than that of the call, reaching a peak of £ 32 400 in year 31, before following the value of the call down to zero during the following nine years. The put follows this pattern because the increasing weight of pension assets invested in highly volatile equities during stage 1 increases the chance of insolvency if there is a sharp downturn in the stockmarket; this risk is reduced during stage 2 as the financial assets are switched into lower risk index bonds. Despite the substantial investment in equities, the objective of ALM is to reduce surplus risk (9) to 0 over the life of the scheme. Fig. 15 shows how this is achieved. Its pattern for the first 31 years is related to the way in which the duration gap is reduced to 0, although for the first eight years the effect of building up an investment in equities without much immediate

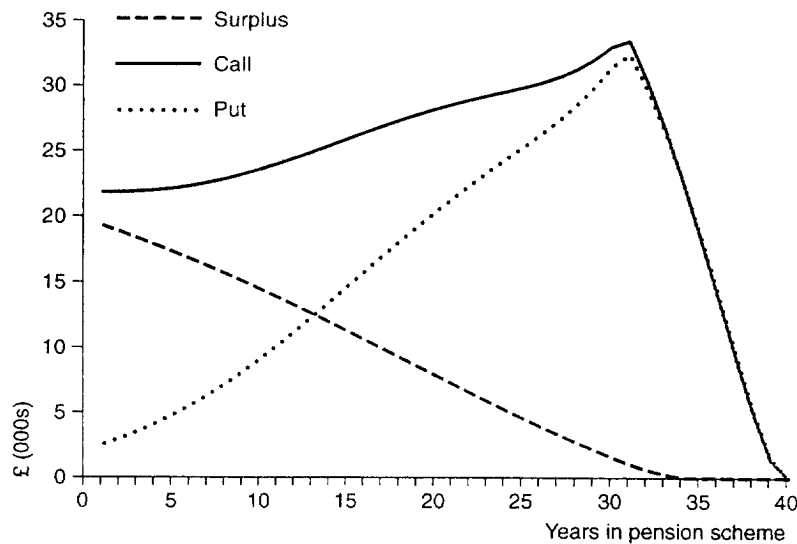


Fig. 14. Pension fund surplus and value of call and put options with minimum contributions.

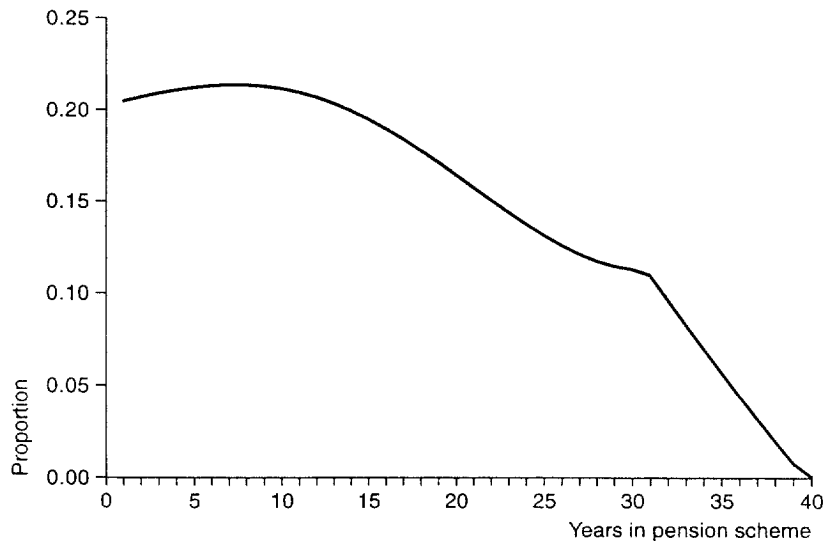


Fig. 15. Surplus risk with minimum contributions.

impact on reducing duration is to raise the surplus risk. Thereafter the duration gap can be preserved at 0 while the LIP accumulates. This takes eight years to achieve and surplus risk falls linearly during this period.

When, in order to provide a cushion of safety, higher than the minimum level of contributions are made to the scheme, then it is possible to add the third stage to the ALM strategy, and this may run in parallel with the first or second stages rather than follow them. We review the case when the member's contribution rate is 5.5%. Fig. 16 shows how the excess financial assets accumulate over the period. It is from these assets that an optimal portfolio involving equities, index bonds and conventional bonds can be constructed on the basis of (33) together with the member or sponsor's degree of risk aversion β . Fig. 17 shows the portfolio weights for the three asset categories for all three stages combined. Comparing with Fig. 12, we see that the asset allocation is still dominated by the requirements of stages 1 and 2 first to build up duration on the asset side and then, once this is equal to that of the

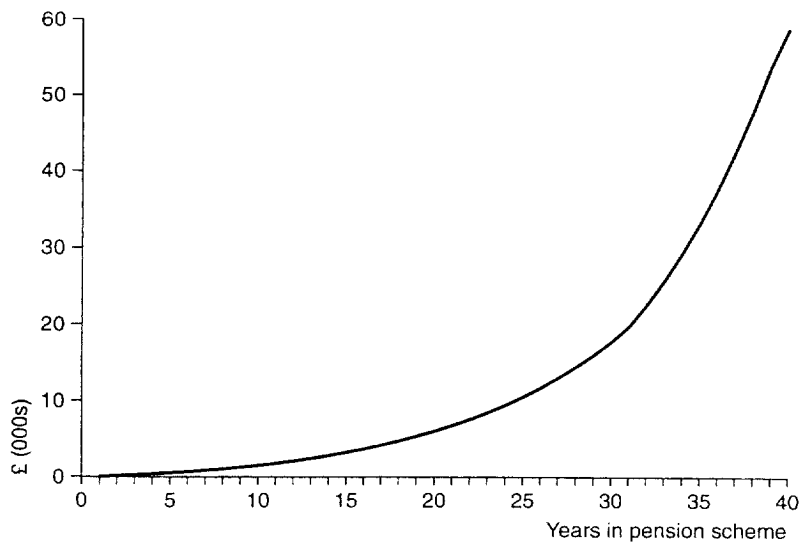


Fig. 16. Excess financial assets with higher contributions.

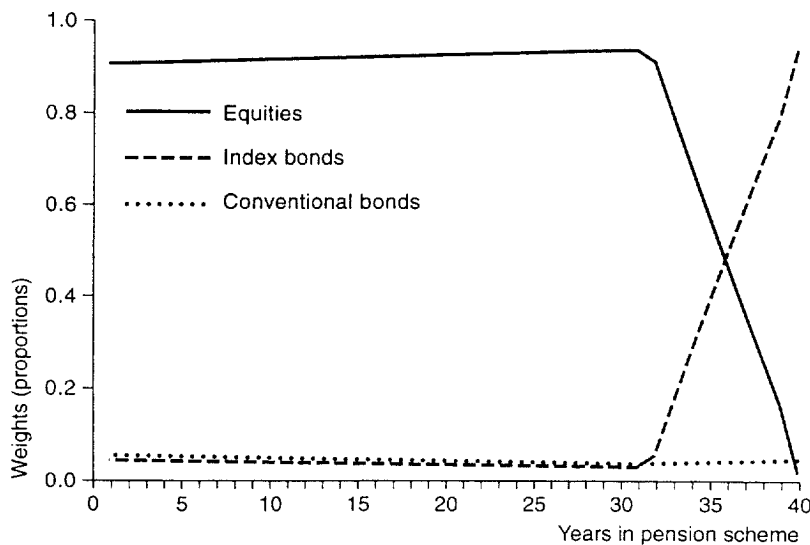


Fig. 17. Financial asset composition with higher contributions.

liabilities, to switch into the LIP. Nevertheless, there is a small role for conventional bonds during the third stage to improve risk diversification: the average holding throughout the period is 4.3%. Fig. 18 shows the combined effect of backloading and excess funding on both the surplus and call and put values. The effect of backloading is to pull the surplus down and this effect dominates in the early period, but eventually the influence of the excess financial assets begins to dominate and helps to push the surplus up to £ 54 500 just prior to retirement: the sponsor exercises the call option and withdraws the surplus just as the member retires. The pattern to the call value is explained by the build up of high-yielding equities during stage 1, followed by the rapid switchover into lower-yielding index bonds combined with the continued investment in equities during stage 2.

Finally, Fig. 19 shows the probabilities of scheme insolvency under different circumstances. In the case of a single-member scheme with a minimum contribution rate of 4.86%, the probability of scheme insolvency remains

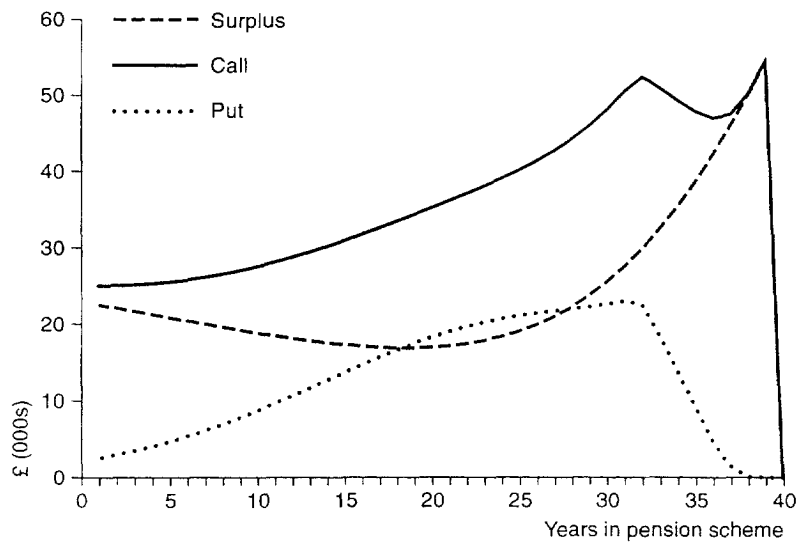


Fig. 18. Pension fund surplus and value of call and put options with higher contributions.

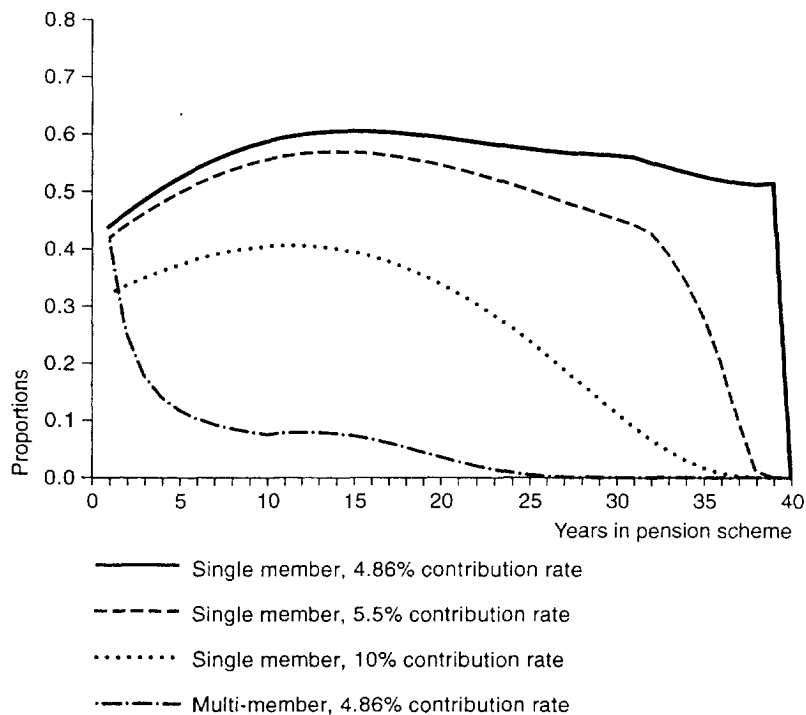


Fig. 19. Probability of scheme insolvency.

high (above 50%) throughout the life of the scheme. The insolvency probability falls as the contribution rate rises, but even a contribution rate of 10% still results in a high insolvency probability until well into the second half of the scheme's existence. However, the insolvency probability falls dramatically in multi-member schemes¹⁶, even

¹⁶ We assume one new member joins the scheme every year.

if only the minimum contribution rate is made into the scheme. This results from the diversification effects from having a portfolio of members at different stages in the same scheme.

7. Conclusion

We have shown that different pension schemes can be treated as different combinations of put and call options on the underlying assets in the scheme with exercise prices related to the value of the liabilities. For example, a defined benefit scheme is equivalent to a defined contribution scheme *plus* a put option (issued by the sponsor) *minus* a call option (issued by the member). This has important implications for the management of pension fund assets. Because the option values depend on both the size and volatility of the pension fund surplus, it is natural for pension fund managers to wish to manage both the surplus and surplus risk using the technique of asset-liability management. ALM has to be implemented as a dynamic strategy of continuous immunisation, involving dynamic reallocations of the asset portfolio between equities, index bonds and conventional bonds as the duration of the liabilities changes. Initially, the optimal strategy is to build up duration by investing in equities. Once the duration of the pension assets equals that of the liabilities, then the optimal strategy is continuous immunisation using a liability immunising portfolio of financial assets. For an unchanged degree of risk aversion, this will lead to a regular rebalancing of the portfolio away from equities and towards index bonds and then, if there are excess assets, towards conventional bonds.¹⁷ The option properties can be used to assess both the appropriateness of the funding level and the effectiveness of the fund management strategy by providing an estimate of the probability of scheme insolvency. Scheme sponsors who are concerned that this probability is too high can reduce it to any desired level by raising the contribution rate into their scheme by a suitable amount.

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¹⁷ This compares with the dynamic reallocations of the asset portfolio between equities and cash proposed by Leland and Rubinstein (1981), and would be as simple to implement as, say, the constant proportion portfolio insurance strategy of Perold and Sharpe (1988) and Black and Perold (1992). For an earlier approach to pension fund immunisation, see Keintz and Stickney (1980).

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