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One-Year Value-At-Risk For Longevity And Mortality

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ONE-YEAR VALUE-AT-RISK FOR LONGEVITY AND MORTALITY

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Abstract
Upcoming new regulation on regulatory required solvency capital for insurers will be predominantly based on a one-year Value-at-Risk measure. This measure aims at covering the risk of the variation in the projection year as well as the risk of changes in the best estimate projection for future years. This paper addresses the issue how to determine this Value-at-Risk for longevity and mortality risk. Naturally this requires stochastic mortality rates. The last decennium a vast literature on stochastic mortality models has been developed. However, very few of them are suitable for determining the one-year value-at-risk. This requires a model for mortality trends instead of mortality rates. Therefore, we will introduce a stochastic mortality trend model that fits this purpose. The model is transparent, easy to interpret and based on well known concepts in stochastic mortality modeling. Additionally, we introduce an approximation method based on duration and convexity concepts to apply the stochastic mortality rates to specific insurance portfolios.

JEL classification: G22; G23; J11

Subject classification: IM10; IE43; IB10

Keywords: one-year value-at-risk, stochastic mortality trend model, Solvency 2

1. Introduction

In recent years there has been an increasing amount of attention of the insurance industry for the quantification of the risks that insurers are exposed to. Important drivers of this development are the increasing internal focus on risk measurement and risk management and the introduction of Solvency 2 (expected to be implemented around 2012).

Solvency 2 will lead to a change in the regulatory required solvency capital for insurers. At this moment this capital requirement is a fixed percentage of the mathematical reserve or the risk capital. Under Solvency 2 the so-called Solvency Capital Requirement (SCR) will be risk-based, and market values of assets and liabilities will be the basis for these calculations.

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Also for pension funds, a new solvency framework will be developed, either as part of Solvency 2 or as a separate project (usually named IORP 2). It is expected that the general principles will be similar as Solvency 2, implying market valuation of assets and liabilities and risk-based solvency requirements.

The SCR will be based on a one-year Value-at-Risk (VaR) measure, corresponding to the 99,5% percentile. This VaR measure aims to cover not only the risk of variation in the projection year, but also the risk of changes in the best estimate projection for future years.

Important risks to be quantified are mortality and longevity risk. Not only is this an important risk for most (life) insurers and pension funds, the resulting solvency requirement will also be part of the market value reserve. Reason for this is that it is becoming best practice to quantify a risk margin (to be included in the value of liabilities) by applying a Cost of Capital rate to the solvency capital necessary to cover for unhedgeable risks, such as mortality and longevity risks.

Börger (2010) provides a good discussion on the model requirements and suitability of current stochastic mortality models for determining the one-year VaR for longevity (and thus mortality) risk. The one-year risk consists of two components:

- the risk that next year's realized mortality will be below (or above) its expectation
- the risk of a decrease (or increase) in expected mortality beyond next year

The first component is the ordinary stochastic variation around the 'best estimate' projection. The second component reflects the risk of a change in the best estimate projection for future years. A cure for cancer is a classical example for this risk. It would take some time before such a new medicine would be available for such a large group of people that mortality for the whole population would be affected. That means that a large effect on next year's realized mortality is not expected, but the impact on future mortality rates can be significant. Therefore, to adequately quantify the VaR for longevity and mortality risk both components have to be addressed properly.

There is a vast literature on stochastic modeling of mortality rates. Most of the stochastic mortality models are so-called spot models that only model the realized mortality. Examples of this are for example Lee and Carter (1992), Renshaw and Haberman (2006), Cairns et al (2006a, 2009) and Plat (2009). For projection purposes, these models contain a mortality trend assumption. However, in most models this trend is fixed and scenarios of realized mortality are derived as random deviations from this trend. This means that those models do not account for the second component of the longevity or mortality risk. This can be overcome in a one-year VaR calculation by generating thousands of Monte Carlo simulations for the next year, treating the simulated mortality rates as a new observation, repeat the calibration process of the spot mortality model, and then project the (fixed) trend forward for each simulation. However, this means that the spot mortality model has to be calibrated thousands of times, while for the projection of the trend in each simulation again simulations would be necessary if it is required to do this precisely. Furthermore, the VaR for the second component would be based on the tail of the distribution of mortality *rates*, not mortality *trends*. Finally, there is a possibility that the risk is underestimated with this approach, because the impact of the next year's realized mortality rates on the calibration of the spot model can be relatively low, depending on the number of historical years underlying the calibration of the model.

The models of Cox et al (2009) and Sweeting (2009) try to solve this issue by allowing for trend changes in the models of Lee and Carter (1992) and Cairns et al (2006a). Both do not account sufficiently for possible changes in trend though, see Börger (2010).

That means that currently the only models that are suitable for these calculations are the so-called forward mortality models, as proposed by for example Dahl (2004), Miltersen and Persson (2005), Cairns et al (2006b) and Bauer et al (2008, 2009). This class of models requires the expected future mortality rates as input and models changes in this quantity over time. The model of Bauer et al (2008, 2009) and its extension by Börger (2010) is the only forward mortality model that has concrete specifications, so this is the only forward model that is readily available.

The forward mortality models model the changes in the mortality rate curve for specific age cohorts, for example the future mortality rates of people of age x_0 at time t_0. This means that such models have to capture the dynamics of each age cohort over time, while each age cohort also contains all ages > x_0: x_0 at t_0, $x_0 + 1$ at $t_0 + 1$, …, $x_0 + k$ at $t_0 + k$, where $k = w - x_0$ and w is the end age of the mortality table (usually set at age 120). In other words, the mortality curves are modeled ‘diagonally’. Capturing these dynamics requires a complex model. Indeed, the model setup of Bauer et al (2008, 2009) is quite complex and not very transparent. This observation, in combination with the fact that the results are (of course) obtained per age cohort, makes the results difficult to interpret. Furthermore, the calibration procedure (given in Börger (2010)) is complex.

Note that insurance companies have to model the mortality rates of males and females simultaneously, adequately addressing the dependence between those. Including this in the forward mortality models would even double the complexity, at least.

Forward mortality models are designed this way to allow for a ‘risk neutral’ specification of the mortality model (for pricing) that can be calibrated to mortality hedging instruments such as longevity bonds. However, currently there is no liquid market for these derivatives, implying that there is no unique risk neutral probability measure (see Cairns et al (2006a)). More importantly, for calculation of the one-year VaR only a ‘real world’ setting is relevant. This observation provides an argument to look for other ways to model stochastic mortality trends, which we will address in this paper.

In this paper a new stochastic mortality trend model is proposed. The trend is represented by a well-known simple reduction factor l_x per age (‘horizontally’). This trend is estimated on subsequent blocks of 30 years of historically observed mortality rates, beginning with 1950-1979, then 1951-1980 and so on. The result of this is a matrix of age by year (per gender), filled with historical observations of (horizontal) mortality *trends*, represented by l_x. Since this form of input is similar as the usual format of historically observed mortality rates and the stochastic mortality trends are also driven by changes in mortality rates, techniques can be applied that are known from the substantial literature of spot mortality models. Concretely, we will use a 3-factor version (per gender) of the spot mortality model described in Plat (2009). After fitting the 3-factor model for all historical years for each gender, the resulting time-series of estimated parameters are simultaneously modeled for males and females in the form of a 6-factor time series model. The advantages of this approach compared to the model of Bauer et al (2008, 2009) are that the model and calibration routine are less complex, the results are easier to interpret and the techniques used are well known from the literature on stochastic mortality models and are standard available in statistical software.

When the stochastic mortality trends are obtained, they have to be applied to the insurance portfolios. While this is possible for an example product, it is practically not feasible for insurance companies to do this for all products in their portfolios. Therefore, we also present an approximation based on the concept of duration and convexity, known from the literature and practice on interest rate risk. Given the simulated mortality rates and the 'mortality duration' and 'mortality convexity', the value of the liabilities can be obtained for each simulation relatively easy.

The remainder of the paper is organized as follows. First, in section 2 the mortality trend is defined and estimated on historical data. Section 3 presents the stochastic model for mortality trends. Section 4 discusses the simulation procedure and section 5 contains a numerical example. Section 6 describes the approximation method based on duration / convexity concepts. Conclusions are given in section 7.

2. Fitting historical mortality trends

The first step in the process is fitting the historical mortality trends. The data used are the historical initial mortality rates of the population of The Netherlands (males and females) for the years 1950-2008 and ages 19.5 – 98.5 [2]. The initial mortality rate q_x is the probability that a person aged x dies within the next year, see Coughlan et al (2007).

A first step is smoothing the age-specific mortality rates across ages to eliminate statistical noise and data errors. This prevents that the projected trends are distorted by this noise. The method used is (penalized) cubic spline smoothing, conform the approach of Coughlan et al (2007). Within this smoothing process the mortality rates are obtained for the ages 20, 21, …98, which will be the basis for the further projection.

The basis for fitting the historical trends will be a relatively simple and well-known deterministic trend model, where the future trend per age is summarized in one parameter:

$$(2.1) \quad q_{x,t} \;=\; l_x \, q_{x,t-1}$$

where t is the year. Using only one parameter for the trend will allow stochastic modeling of the future trends using well known techniques later on in the process. The variable l_x is sometimes also named 'mortality reduction factor'.

[2] Source: http://www.statline.nl

This model is fitted for each age on subsequent blocks of 30 years of historically observed mortality rates, beginning with 1950-1979, then 1951-1980 and so on, until the period 1979-2008. The reason for using blocks of 30 years is that both fitting the historical trends and fitting the stochastic model for these trends will be based on enough data (30 years). For example, when blocks of 40 years would be used, we would only have 20 years of observed historical l_x's. Furthermore, it is consistent with the approach of Börger (2010).

Now one possible approach for fitting the trends is to write (2.1) as

$$(2.2) \quad \ln q_{x,t} - \ln q_{x,t-1} = \ln l_x$$

Then applying least squares estimation gives:

$$(2.3) \quad \ln \hat{l}_x = \sum_{i=1}^{n-1} w_i \, \Delta \ln q_{x,i}$$

where n = 30, w_i = 1/(n-1) and $\Delta\ln(q_{x,i}) = \ln(q_{x,i+1}) - \ln(q_{x,i})$. So $\ln(\hat{l}_x)$ is the average of the observed differences $\Delta\ln(q_{x,i})$.

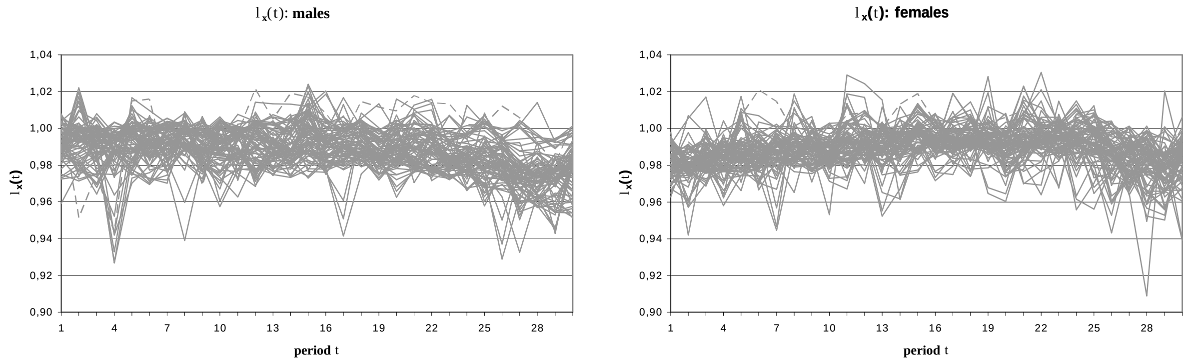
However, when using this approach the change in trend over the years is not only driven by the newly observed year, but also by not using the earliest year of the foregoing block. For example, the change in trend that is observed after fitting the trend for subsequent periods 1950-1979 and 1951-1980, is not only caused by adding the year 1980, but also by not using 1950 in the latest trend fitting process. For the one-year VaR we are more interested in the impact of the latest observation though.

Because of this, we will use a slightly different approach. We will apply an ARIMA(0,1,1) model without constant to the differences $\Delta\ln(q_{x,i})$. This is equal to the well-known 'Exponential Smoothing' technique, where the most recent observations are weighted more than the earlier observations, e.g. $w_j > w_i$ for $j > i$. Consequently, the earliest observation is weighted the least of all observations. This approach will therefore limit the impact of excluding the earliest observation. Another reason for applying this process is the continuously accelerating downward trend in observed mortality rates of males in The Netherlands, which has lead to practitioners using models that weight the recent trends most. Also, it allows for a possible different treatment of the short / middle term and the long term trend (see paragraph 4.2).

The ARIMA(0,1,1) model is fitted for each age for the 30 subsequent periods, leading to a matrix of age by year (per gender), filled with historical observations of mortality trend $l_x(t)$, where t is the indicator for the subsequent periods of data, i.e. the period 1950-1979 is denoted by t = 1, 1951-1980 by t = 2, and so on. Note that this data structure is similar as the usual format of historically observed mortality rates. Since the fitted trends $l_x(t)$ are also driven by changes in mortality rates, techniques can be applied that are known from the substantial literature of spot mortality models.

Figure 2.1 shows the estimated mortality trends $l_x(t)$ for males and females and all ages, based on the data of The Netherlands. Each line represents the time series of l_x for a specific age *x*.

Figure 2.1: estimated mortality trends** $l_x(t)$ **for males and females, all ages



The figures show that the historically observed $l_x(t)$'s are concentrated between 0,98 and 1, indicating that the trend in mortality rates is usually downwards. Furthermore, the acceleration of the downward trend for males is visible between periods 20 – 30.

Further characteristics of the time series are given in table 2.1 for ages 25, 45, 65 and 85. Note that the absolute differences might seem small. However, the impact on the value of the liabilities can be very significant, since these $l_x(t)$'s are applied for a long future period of years.

Table 2.1: characteristics estimated mortality trends

	Average				Standard
Age group	**t Î (1-10)**	**t Î (11-20)**	**t Î (21-30)**	**total**	**deviation**
Males					
age 25	0,987	0,989	0,981	0,986	0,014
age 45	0,981	0,988	0,979	0,983	0,011
age 65	0,992	0,984	0,971	0,982	0,010
age 85	0,999	1,001	0,990	0,996	0,008
Females					
age 25	0,986	0,994	0,983	0,988	0,014
age 45	0,987	0,997	0,994	0,992	0,014
age 65	0,985	0,994	0,986	0,988	0,007
age 85	0,986	0,993	0,989	0,989	0,006

The table shows different trends, trend patterns and standard deviations for younger ages (age 25 and 45), age 65 and age 85. Also, differences between males and females are clearly visible. The stochastic model for projecting future mortality trends should capture this different behavior between ages and gender.

3. A stochastic model for mortality trends

The next step in the process is defining a stochastic model for the mortality trends. First, a parametric model across ages will be fitted on the yearly observations of l_x, in line with concepts from spot mortality models. To the resulting time series of fitted parameters a suitable time series model has to be applied.

3.1 Fitting a parametric model per year

The mortality trends $l_x(t)$'s estimated in the previous section are, of course, driven by the underlying mortality rates. Therefore, we can apply concepts and techniques from the literature on stochastic spot mortality models. The results in Plat (2009) indicate that at least 3 stochastic factors are required to model the dependence between ages adequately. This can also be concluded from table 2.1 in the previous section, where we observe differences between young, middle and old ages. Therefore, we define a 3-factor model (per gender), based on the model structure in Plat (2009):

$$(3.1) \quad l_x(t) = k^1(t) + k^2(t)\left(\bar{x} - x\right) + k^3(t)\left(\bar{x} - x\right)^+$$

where $\left(\bar{x} - x\right)^+ = \max\left(\bar{x} - x, 0\right)$. The model has 3 stochastic factors but has a relatively simple structure. For countries where a clear cohort effect in the historical mortality rate observations is observed, cohort parameters could be added to (3.1), see Plat (2009).

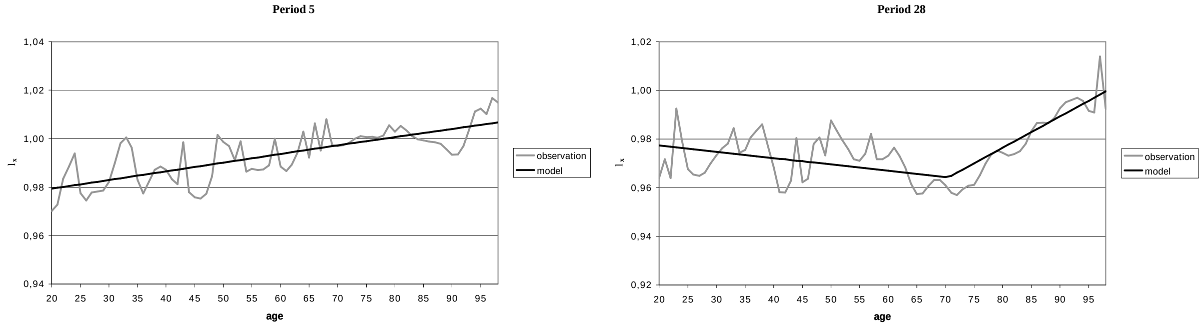
The factor k^1 represents the base level of the mortality trend and allows for changes in mortality trends for all ages simultaneously. The factor k^2 allows changes in mortality trend to vary between ages, to reflect the observation from the previous section that the change in mortality trend can differ for different age classes. The factor k^3 is added to capture the specific dynamics of younger ages. The parameter $\bar{x}$ is a constant that is also estimated from the data.

The factors k^2 and k^3 allow the model to have a non-trivial correlation structure between ages. How the three factors k^1, k^2 and k^3 interact exactly will be different for each population / gender, depending on the patterns observed in the historical observations.

De parameters of model (3.1) can be determined using standard Maximum Likelihood estimation. This estimation procedure can be done using standard functionality of statistical software (such as SAS, R or Matlab).

Figure 3.1 shows the fit of the model for two random periods. Despite the simple linear structure of the model the fit to the observed data is good. Note that this structure does not mean that eventually the stochastic l_x's are all of this shape; this will be explained further in section 4.

Figure 3.1: fit of the model for period 5 and 28, males



The fitting procedure described above leads to time series of estimations of k^1, k^2 and k^3. These are given in figure 3.2 (males) and figure 3.3 (females). The next step in fitting the model is selecting and fitting a suitable model to these time series.

Figure 3.2: time series of estimated k's – males

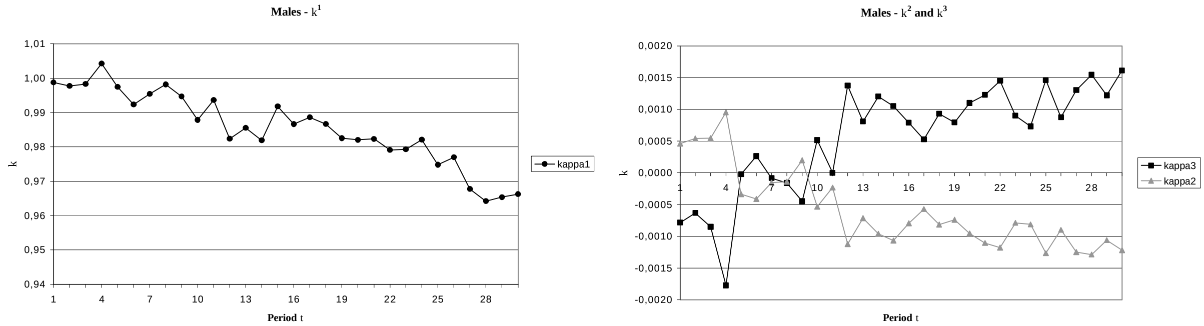
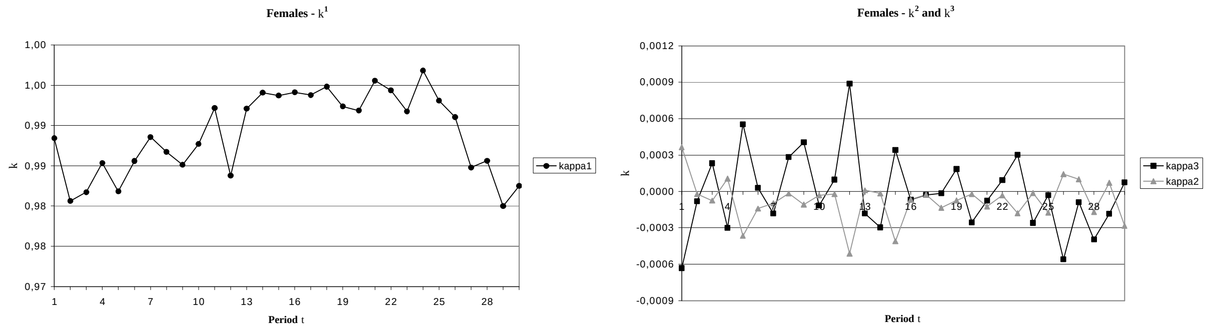


Figure 3.3: time series of estimated k's - females



3.2 Selecting a suitable time series model

The most recent l_x is based on the period 1979 – 2008 and provides the 'best estimate' projection of mortality rates per ultimo 2008. The aim of the proposed stochastic model is to quantify the stochastic variation around this best estimate projection of mortality rates. Since the most recent l_x already captures the best estimate trend, the time series model for mortality trends l_x should be a zero-trend process. Note that we smooth this most recent l_x (again using a cubic spline) to avoid illogical patterns over ages due to statistical noise (as can be noticed in figure 3.1).

Since insurance portfolios are exposed to longevity and mortality exposure of males and females, it is important to model both genders and its dependence adequately. Therefore, we model both males and females simultaneously in a 6-factor time-series model. To come to a specification of this 6-factor model, we have first fitted ARIMA models (see Box and Jenkins (1976) or Verbeek (2008)) to the univariate time series of the estimated k's. Since for some populations there can be a long term relationship between k^1 on the one hand and k^2 and k^3 on the other hand, we also tested all possible models for k^2 and k^3 including k^1 as explaining variable. The models with the most preferable value for the Bayes Information Criterion (*BIC*) (see Verbeek (2008))[3] are selected. The combination of these models is used as a first basis for the 6-factor model, from where alternative specifications (for example, with less parameters) were tested based on the estimation results of the combined model and its parameters. This leads to the following selected processes for the k's (m = males, f = females):

- $k_m^1(t)$: ARIMA(1,1,0), no constant
- $k_m^2(t)$ and $k_m^3(t)$: ARIMA(0,1,0), no constant, $k_m^1(t)$ as explaining variable
- $k_f^1(t)$ and $k_f^2(t)$: ARIMA(1,1,0), no constant
- $k_f^3(t)$: ARIMA(0,1,0), no constant, $k_f^1(t)$ as explaining variable

This can be written as the following multivariate model:

$$
\begin{aligned}
\text{(3.2)}\quad k_m^1(t) &= k_m^1(t-1) + b_{1,m}\left[k_m^1(t-1) - k_m^1(t-2)\right] + e_m^1(t) \\
k_m^2(t) &= k_m^2(t-1) + a_{2,m}\left[k_m^1(t) - k_m^1(t-1)\right] + e_m^2(t) \\
k_m^3(t) &= k_m^3(t-1) + a_{3,m}\left[k_m^1(t) - k_m^1(t-1)\right] + e_m^3(t) \\
k_f^1(t) &= k_f^1(t-1) + b_{1,f}\left[k_f^1(t-1) - k_f^1(t-2)\right] + e_f^1(t) \\
k_f^2(t) &= k_f^2(t-1) + b_{2,f}\left[k_f^2(t-1) - k_f^2(t-2)\right] + e_f^2(t) \\
k_f^3(t) &= k_f^3(t-1) + a_{3,f}\left[k_f^1(t) - k_f^1(t-1)\right] + e_f^3(t)
\end{aligned}
$$

where the b parameters are used for autoregressive terms, the a parameters for the explaining variables. The e's are the error terms with covariance matrix S.

The parameters of this model can be estimated by any technique suitable for estimating simultaneous systems of equations. We use Full Information Maximum Likilihood (FIML), see Chernoff and Divinsky (1953). The estimated parameter values are: $b_{1,m}$ = -0,311 , $a_{2,m}$ = 0,065 , $a_{3,m}$ = -0,097 , $b_{1,f}$ = -0,371 , $b_{2,f}$ = -0,281 , $a_{3,f}$ = -0,121.

In section 1 it is mentioned that it's important to model the mortality trends of males and females simultaneously to adequately capture the dependence between those. In that respect, it is

[3] The measure *BIC* provides a trade-off between fit quality and parsimony of the model.

interesting to see the estimated correlation matrix between the k-processes. These are given in table 3.1.

Table 3.1: estimated correlation matrix of multivariate model

	k_m^1	k_m^2	k_m^3	k_f^1	k_f^2	k_f^3
k_m^1	1,00	-0,24	0,15	0,43	0,18	0,40
k_m^2		1,00	-0,96	-0,14	0,54	-0,53
k_m^3			1,00	0,12	-0,59	0,52
k_f^1				1,00	0,37	0,46
k_f^2					1,00	-0,43
k_f^3						1,00

The table shows that the correlation between the k's for males and the equivalent k's for females is in a range of 0,4 - 0,55. If one would not model the mortality trends of males and females simultaneously and for example simply aggregate the VaR's (implicitly assuming full correlation) the dependence would be significantly mis-specified.

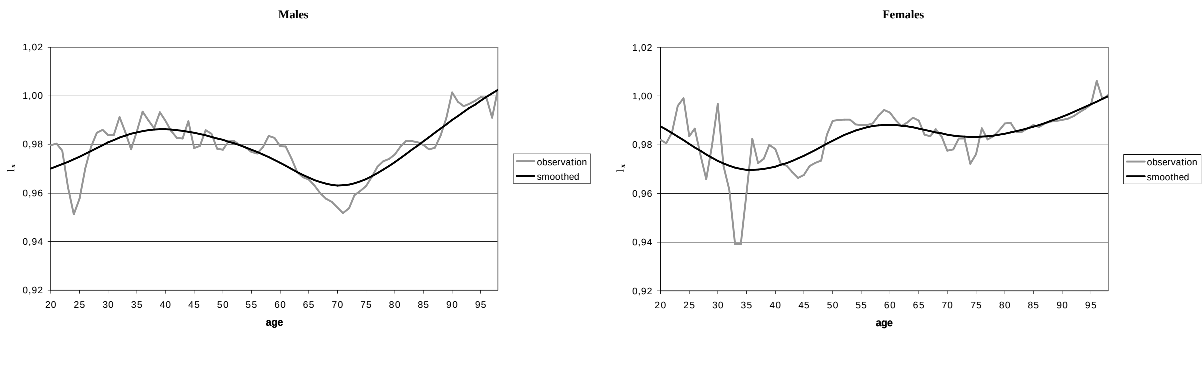
4. Simulation procedure

Using the model defined in section 3 and its estimated parameters, the distribution of mortality trends for a one-year horizon can be determined using Monte Carlo simulation. Paragraph 4.1 describes the simulation procedure. Paragraph 4.2 introduces a possible extension to the procedure, which allows for a different treatment of the short / medium term and long term trend.

4.1 Simulation procedure

As mentioned in paragraph 3.2 the first step is smoothing the most recent l_x which is the basis for the best estimate projection. This should give a realistic pattern of l_x between ages, and avoids statistical noise. Again (penalized) cubic spline smoothing is used for this. The most recently observed and smoothed l_x's for males and females are given in figure 4.1.

Figure 4.1: observed and smoothed l_x's (2008), males and females



The simulation procedure consists of the following steps:

a) **Simulate *k*'s** Draw a random sample from the multivariate model (3.2) for the *k*'s.
b) **Determine l_x's** Given the simulated *k*'s, the l_x's for each age can be determined for each simulation.
c) **Translate l_x in factor** Divide the simulated l_x's by the most recent l_x's (in this case per ultimo 2008) resulting from model (3.1) to determine a factor reflecting the relative change.
d) **Obtain 'new' l_x's** Apply the factor determined in step c) to the most recently observed and smoothed l_x's (as given in figure 4.1). This gives the simulated l_x's similar smooth shapes as in figure 4.1 and ensures that the simulated l_x's in the projection year have the same starting point as the best estimate projection.
e) **Determine corresponding q_x's** To come to a projection of mortality rates q_x in each simulation, not only the mortality trends l_x are required but also the (simulated) q_x's in the projection year. These are determined using the following formula:

$$(4.1) \qquad \frac{q_{x,t} - l_x(t-1)\, q_{x,t-1}}{l_x(t-1)\, q_{x,t-1}} = q_x \, \frac{l_x(t) - l(t-1)}{l(t-1)}$$

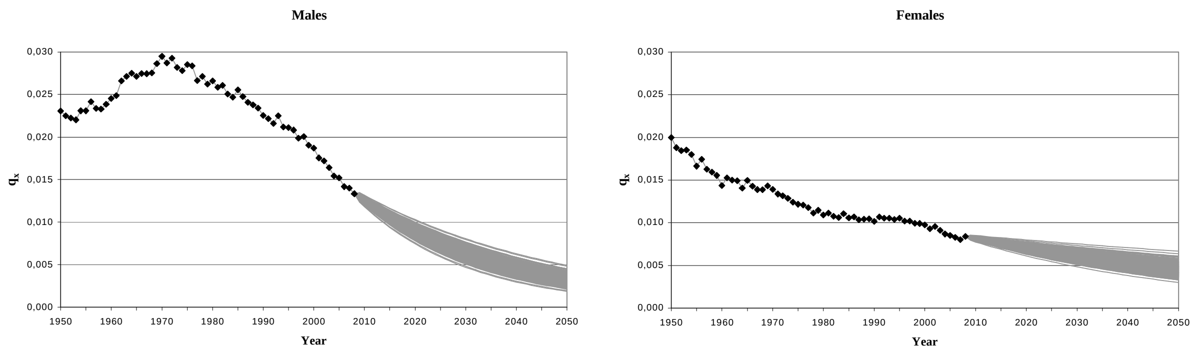
In words, this means that the relative change in q_x (compared to its expectation) is determined by applying a constant q_x per age to the relative change in l_x. The constant q_x is determined per age by dividing the historical standard deviation of q_x by the historical standard deviation of l_x. The resulting q_x's are smoothed over ages using a (penalized) cubic spline. This approach ensures that the volatility of simulated q_x's is in line with historically volatility of mortality rates (per age) and establishes the relationship between l_x and q_x.

f) **Determine projected q_x's** Based on the l_x's from step d) and the q_x's from step e), the projected q_x's for the next, say, 100 years can easily be determined using (2.1).

Based on the simulations of projected mortality rates in step f) the one-year VaR for longevity and mortality risk can be determined by calculating the value of the liabilities in each simulation. An example of this is worked out in the next section.

Figure 4.2 shows a sample of 250 simulations for the projected mortality rate q_x (as determined in step f)) for age 65 for the period 2009-2050, based on a horizon of 1 year. The black line gives the historical observed mortality rate up until 2008.

Figure 4.2: 250 simulations of projected q_x for age 65, males and females



4.2 Possible extension: introducing difference in short / medium and long term trends

Different specifications for the calibration of model (2.1) are possible. For example, one could argue that it is not necessarily the case that a (simulated) trend will hold on for the next 50 years. It could be that there is a specific reason (for example, smoking) that impacts the mortality trend for several years, after which the trend is normalizing again. Therefore, a possible addition to the model presented in this paper could be to distinguish the short term and the long term trend. As mentioned in section 2, the l_x's in this paper are fitted using Exponential Smoothing, which means that more recent observations are more heavily weighted. One can assume that these l_x's are suitable for projection of mortality rates for the short and medium term, say up to 15 years. Furthermore, since the fitting of model (2.1) is based on 30 years of historical data, it implies that a long term target trend is necessary for projections for year 30 and further. Therefore, we can include a (deterministic) long term trend based on a longer period of historical data using (2.3). Additionally, we can assume that for each simulation the l_x's grow from the simulated value in year 15 to the long term target in year 30. The only additional decision to make is on what historical period to base the long term trend. A general viewpoint could be to use all available data for estimating the long term trend (in this case 1950 – 2008). This more or less justifies the assumption that the long term trend is deterministic, since the impact of a new observation on a trend fitted on such a long period of historical data (weighted equally) will not be substantial.

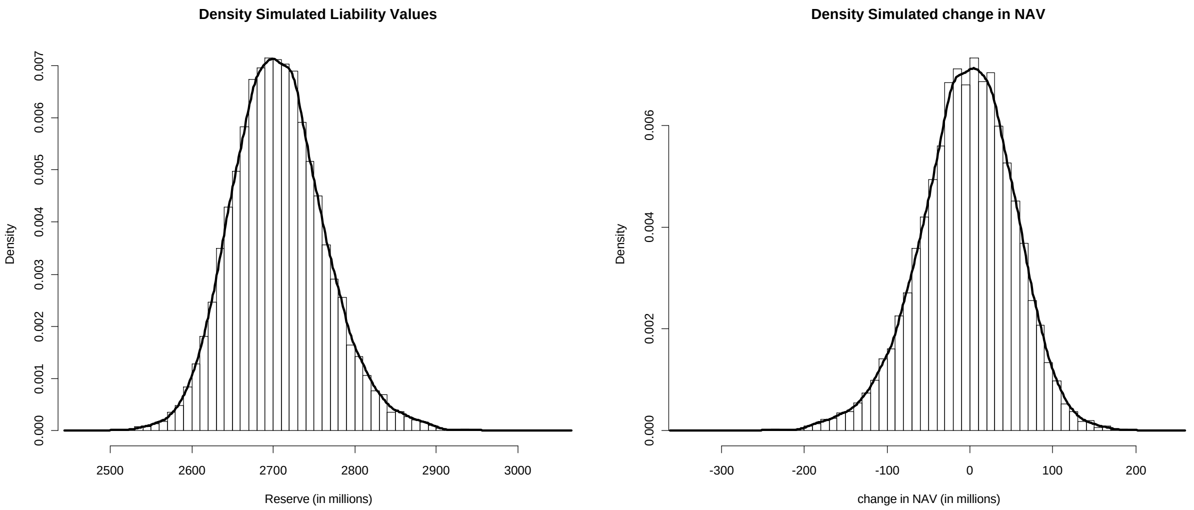
Note that including a deterministic long term trend level does not mean that no risk for cash flows for year 30 and further is quantified. The simulated l_x's will impact the level of mortality rates for the full runoff of the portfolios.

5. Numerical example

The simulation procedure as described in section 4 results in the simulations of projected mortality rates. Based on these simulations, the one-year VaR for longevity and mortality risk can be determined by calculating the value of the insurance liabilities in each simulation. In this section a numerical example is worked out for an annuity portfolio of a European insurer[4]. The portfolio consists of 45.000 male and 36.000 female policyholders of age 65 or older. Note that this portfolio is only exposed to longevity risk, not mortality risk.

Figure 5.1 shows the densities of the value of the insurance liabilities and the change (delta) in Net Asset Value (NAV) of the company. Note that the delta of the NAV is just the opposite of the movement in the value of the insurance liabilities. The results are based on 10.000 simulations.

Figure 5.1: densities of value of insurance liabilities and delta Net Asset Value (NAV)



The resulting VaR's for different confidence levels are given in table 5.1 for males, females and the total portfolio. The VaR's are also given as percentage of the best estimate of liabilities of the (sub-)portfolio (between brackets).

Table 5.1: VaR for different percentiles, nominal and as % of best estimate liabilities (between brackets)

Value-at-Risk	Confidence Level					
(€ millions)	75%	90%	95%	99%	99,50%	99,95%
Males	38,7 {1,4%}	76,1 {2,8%}	100,9 {3,7%}	154,0 {5,7%}	172,9 {6,4%}	234,2 {8,7%}
Females	66,7 {3,0%}	99,6 {4,4%}	120,4 {5,3%}	157,8 {7,0%}	174,0 {7,7%}	216,7 {9,6%}
Total	98,7 {2,0%}	160,5 {3,2%}	201,1 {4,1%}	273,4 {5,5%}	298,5 {6,0%}	375,7 {7,6%}

The table shows that the uncertainty for the females is higher than for the males, relative to its best estimate. Furthermore, the diversification effect between males and females is clearly visible, since the total VaR is less than the sum of the VaR's of males and females. The impact of diversification is larger for higher confidence intervals.

[4] The extension introduced in paragraph 4.2 is not included in this example. Including this extension would result in a lower Value-at-Risk.

Within the Solvency 2 framework a standard formula is defined for quantifying the VaR (or SCR), corresponding to the 99,5% percentile on a one-year horizon. The VaR for longevity risk is based on a shock downwards of 25% for all mortality rates. Table 5.2 compares the results of applying this shock with the VaR's reported in table 5.1.

Table 5.2: Comparison 99,5% VaR with Solvency 2 standard formula

(€ millions)	99,5% VaR	Solvency 2	Difference
Males	172,9	244,3	-29%
Females	174,0	165,1	+ 5%
Total	298,5	409,4	-27%

The table shows that the VaR for the total portfolio using the proposed model in this paper is 27% lower than the standard formula of Solvency 2. This difference is mainly caused by the lower VaR for male policyholders compared to Solvency 2.

The 99,5% VaR's in table 5.1 can be back solved to an implied shock for all mortality rates, consistent with the definition of the standard formula of Solvency 2. For males, females and the total this leads to implied shocks of respectively -18,3%, -26,2% and – 18,9%.

Of course, the results above are specific for the Dutch population and the insurance portfolio used. For other populations or portfolios, the relative results of males / females and VaR's / Solvency 2 can be very different.

6. Duration / Convexity approach

When the stochastic mortality trends are obtained, they have to be applied to the insurance portfolios. While this is possible for an example product such as in section 5, it is practically not feasible for insurance companies to do this for all products in their portfolios. The reason for this is that insurance companies usually model the future liabilities in detail per policy, potentially leading to runtimes of a few hours per scenario.

Another approach that is sometimes used is to pick the particular percentile from the distribution of mortality rates and calculate the value of the liabilities for the selected scenario. A different scenario has to be picked for mortality and longevity risk, and these are aggregated using a correlation parameter. For products that contain both mortality and longevity risk, the mortality and longevity scenarios are applied separately and results are again aggregated using a correlation parameter. The standard formula of Solvency 2 also uses a version of this 'shock'-approach. The problem with this approach is that insurance portfolios consist of a combination of products with mortality risk, products with longevity risk and products with both risks. That means that it is generally not clear what the 'true' percentile of the total portfolio is until the portfolios are actually confronted with the stochastic mortality scenarios. As such, picking percentiles and setting a correlation parameter will generally not capture the dependence between the insurance products and longevity and mortality risks adequately.

Therefore, there is a need for an adequate approximation of the value of the insurance liabilities, given a simulated path of mortality rates. Inspired by interest rate risk theory (see for example Fabozzi (2006)), in this paragraph an approximation based on duration and convexity is discussed. This has been tested before by Beckers (2010) on some extreme mortality scenarios.

Coughlan et al (2007) introduced the term '*q*-duration' to denote the sensitivity of the value of insurance liabilities for a change in future mortality rates. We will stick to this naming convention and name the mortality convexity measure '*q*-convexity'. The *q*-duration and *q*-convexity measures can be implemented in several ways (see for example Li and Hardy (2009) or Wang et al (2009)). In this paper we use the following formulas for obtaining the 'effective' *q*-duration and *q*-convexity:

$$(6.1) \quad q-duration = \frac{MVL_{-} - MVL_{+}}{2\left(MVL_0\right)\Delta q}$$

$$(6.2) \quad q-convexity = \frac{MVL_{+} + MVL_{-} - 2MVL_0}{\left(MVL_0\right)\left(\Delta q\right)^2}$$

where Δq is the change in all mortality rates (in terms of percentage), MVL_0 represents the market value of insurance liabilities and MVL_- and MVL_+ denote this value for respectively a decrease and increase of the mortality rates by Δq.

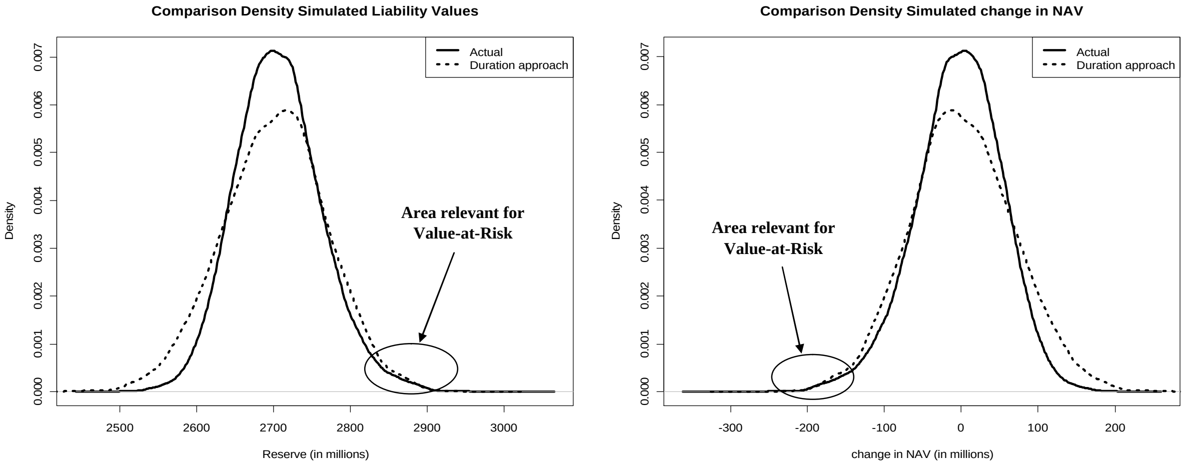
Given a shock in mortality rates of q_{shock}, the approximated value of liabilities after the shock MVL_{shock} can be determined as:

$$(6.3) \quad MVL_{shock} = MVL_0 - qD \times MVL_0 \times q_{shock} + 1/2 \times qC \times MVL_0 \times \left(q_{shock}\right)^2$$

where *qD* and *qC* are the *q*-duration and *q*-convexity. Using this formula, the value of the insurance liabilities can be obtained for each simulation relatively easy. However, in this case the situation is more complex than for interest rates. Concretely, it might be difficult to obtain the simulated q_{shock} , because the change in q_x is different for different ages and future years. Therefore, either the simulated q_x's have to be weighted appropriately or some calibration step has to be done. In this paper the latter approach is used. Since the portfolio only contains policyholders of age 65 or older, the relevant q_x's are in the triangle starting from age 65 at time t_0 until age (65 + *y*) at time (t_0 + *y*). To determine the height of the triangle *y* we have picked one tail scenario (roughly in the expected area of the 99% - 99,5% percentile) from the simulation set and solved for the value of *y* that approximates the liabilities well in this scenario. For this specific portfolio, it turns out that for *y* = 30 gives the best approximation for this tail scenario.

Figure 6.1 shows a comparison of the actual densities of the value of insurance liabilities for male policyholders and the delta Net Asset Value (NAV) presented in figure 5.1 with densities obtained using the approximation described above.

Figure 6.1: comparison actual densities with approximated densities



The figure shows that the duration / convexity approach cannot approximate the full distribution sufficiently. However, the approximation does seem to work well for the area that is relevant for the VaR calculations. The reason for this is that, as mentioned above, the duration / convexity is calibrated to a tail scenario. For the 99,5% confidence interval, the duration / convexity approximation leads to an VaR of € 170,1 million, which is only 1,6% lower than the result in table 5.1. Note that the results in table 5.1 are also not exact, since there is always some simulation error included.

7. Conclusions

With the introduction of Solvency 2 the regulatory capital requirement for insurers will be based on a one-year Value-at-Risk (VaR) measure, corresponding to the 99,5% percentile. This VaR measure aims to cover not only the risk of variation in the projection year, but also the risk of changes in the value of insurance liabilities in that year. This paper concentrated on longevity and mortality risk.

Most of the existing stochastic mortality models are so-called spot models that only model the realized mortality. These models do not account sufficiently for the second component, the change in the value of insurance liabilities, of the longevity or mortality risk. That means that currently the only models that are suitable for these calculations are the so-called forward mortality models, such as presented in Bauer et al (2008, 2009). However, this model setup is quit complex and not very transparent, making the results difficult to interpret. Furthermore, taking into account female mortality rates and its dependence with male mortality rates would double the complexity, at least.

In this paper a new stochastic mortality trend model is proposed. The trend is represented by a simple reduction factor l_x per age ('horizontally'). This trend is estimated on subsequent blocks of 30 years of historically observed mortality rates. The result of this is a matrix of age by year (per gender), filled with historical observations of (horizontal) mortality *trends*. Since this form of input is similar as the usual format of historically observed mortality rates and the stochastic mortality trends are also driven by changes in mortality rates, techniques can be applied that are known from the substantial literature of spot mortality models. We have used a 3-factor version (per gender) of the spot mortality model described in Plat (2009). After fitting the 3-factor model for all historical years for each gender, the resulting time-series of estimated parameters are simultaneously modeled for males and females in the form of a 6-factor time series model. The advantages of this approach compared to the model of Bauer et al (2008, 2009) are that the model is less complex, the results are easier to interpret and the techniques used are well known from the literature on stochastic mortality models and are standard available in statistical software.

When the stochastic mortality trends are obtained, they have to be applied to the insurance portfolios. While this is possible for an example product, it is practically not feasible for insurance companies to do this for all products in their portfolios. Therefore, we proposed an approximation based on the concept of duration and convexity, known from the literature and practice on interest rate risk. Given the simulated mortality rates and the 'mortality duration' and 'mortality convexity', the value of the liabilities can be obtained for each simulation relatively easy. The duration / convexity approach cannot approximate the full distribution sufficiently, but the approximation does seem to work well for the area that is relevant for the VaR calculations.

We have suggestions for further research. First of all, different specifications for the model (2.1) for q_x and (3.1) for l_x are possible. Other specifications can be defined and the impact of it can be compared with that of the model used in this paper. Also, the extension introduced in paragraph 4.2 can be explored further. Furthermore, research can be done to a refinement of the duration / convexity approach, possibly approximating a larger part of the distribution adequately.

References

BAUER, D., M. BÖRGER, J. RUSS AND H. ZWIESLER (2008): The volatility of mortality, *Asia-Pacific Journal of Risk and Insurance 3*, 184-211

BAUER, D., M. BÖRGER AND J. RUSS (2009): On the pricing of longevity-linked securities, *Insurance: Mathematics and Economics 46*, 139-149

BECKERS, E. (2010): Approximation of economic capital for mortality and longevity risk, *AENORM 18*, 36-41

BÖRGER, M. (2010): Deterministic shock vs. stochastic value-at-risk – an analysis of the Solvency 2 standard model approach to longevity risk, *Working paper*

BOX, G.E.P. AND G.M. JENKINS (1976): Time series analysis: forecasting and control, *Holden-Day San Fransisco*

CAIRNS, A.J.G., D. BLAKE, AND K. DOWD (2006a): A two-factor model for stochastic mortality with parameter uncertainty: Theory and Calibration, *Journal of Risk and Insurance 73*, 687-718

CAIRNS, A.J.G., D. BLAKE, AND K. DOWD (2006b): Pricing death: Frameworks for the valuation and securitization of mortality risk, *ASTIN Bulletin 36*, 79-120

CAIRNS, A.J.G., D. BLAKE, K. DOWD, G. D. COUGHLAN, D. EPSTEIN, A. ONG AND I. BALEVICH

(2007): A quantitative comparison of stochastic mortality models using data from England & Wales and the United States, *North American Actuarial Journal 13*, 1-35
COUGHLAN, G., D. EPSTEIN, A. ONG, A. SINHA, I. BALEVICH, J. HEVIA-PORTOCARRERO, E. GINGRICH, M. KHALAF-ALLAH AND P. JOSEPH (2007): Lifemetrics Technical Document, *available at: http://www.jpmorgan.com/pages/jpmorgan/investbk/solutions/lifemetrics*
COX, S., Y. LIN AND A. MILDONIS (2009): Mortality regimes and pricing, *Working paper*
DAHL, M. (2004): Stochastic mortality in life insurance: Market reserves and mortality-linked insurance contracts, *Insurance: Mathematics and Economics 35*, 113-136
FABOZZI, F.J. (2006): Bond markets, analysis and strategies, 6th edition, *Prentice Hall*
MILTERSEN, K. AND S. PERSSON (2005): Is mortality dead? Stochastic force of mortality determined by no arbitrage, *Working paper*
LEE, R.D., AND L.R. CARTER (1992): Modelling and forecasting U.S. mortality, *Journal of the American Statistical Association 87*, 659-675
LI, J. AND M. HARDY (2009): Measuring basis risk involved in longevity hedges, *Working paper*
PLAT, R. (2009): On stochastic mortality modeling, *Insurance: Mathematics and Economics 45*, 393-404
RENSHAW, A.E., AND S. HABERMAN (2006): A cohort-based extension to the Lee-Carter model for mortality reduction factors, *Insurance: Mathematics and Economics 38*, 556-570
SWEETING, P. (2009): A Trend-change extension of the Cairns-Blake-Dowd model, *Working paper*
VERBEEK, M. (2008): Modern Econometrics, 3th edition, *John Wiley & Sons, Ltd*
WANG, J.L., H.C. HUANG, S.S. YANG AND J.T. TSAI (2009): An optimal product mix for hedging longevity risk in life insurance companies: the immunization theory approach, *Journal of Risk and Insurance, forthcoming*