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Inter-temporal Optimization and Deterministic Lifestyle Strategy in Managing Defined-Contribution Pension Plans

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Inter-temporal optimization and deterministic lifestyle strategy in managing defined-contribution pension plans

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Abstract

Inter-temporal optimization and deterministic lifestyle asset allocation strategies for defined-contribution pension plans are investigated and compared both analytically and numerically. The pension plan is assumed to invest in two types of asset, risk free assets and equities, or bonds and equities, and the plan members' terminal utility a power function of pension wealth at retirement with their final wages as numeraire. The optimal asset allocation strategy using two assets is derived analytically for fully hedgeable wage incomes and compared numerically with that for non-hedgeable wage income and deterministic lifestyle strategy. The deterministic lifestyle strategy is shown to be replicable by a static allocation strategy with same expected returns and lower variances. The inter-temporal optimization strategy outperforms the lifestyle strategy in numerical simulations both when there is no further pension contribution or non-hedgeable wage risk and when the wage income is not fully hedgeable. When there are further pension contributions, the optimal proportion invested in the more risky asset is higher than that when future pension contributions are transformed into augmented wealth by short-selling a replicating portfolio to be paid by future pension contributions. With usual assumptions on market parameters, the optimal pension portfolio composition is independent of the value of non-hedgeble wage risk and the value of pension contribution rate.

Keywords: Optimal asset allocation; Defined-contribution pension plan; Lifestyle; Power utility; Hamilton-Jacobi-Bellman equation; wage risk.

1. Introduction

It appears to be a received wisdom that for a defined-contribution (DC) pension plan more weight should be put into the potentially high-return, high risk stocks (equities) in the early years of the pension plan in order to increase the pension wealth, whereas more weight should be put into the riskless or low risk assets in the final years of the pension plan to avoid a sudden change in stock market eroding away its wealth. The "reason" given by people for doing this is that pension wealth may recover from a fall in stock value early in the pension plan, while a fall later may leave no time for the pension wealth to recover. Investment strategies guided by such thoughts have been termed as lifestyle strategies.

Although the lifestyle asset allocation strategy is popular with financial advisors and pension fund managers, the thinking behind the simple deterministic lifestyle strategy has been contested by academic studies. Samuelson (1969) showed that a rational maximizer of expected utility, with constant relative risk aversion and facing random-walk securities returns, would rationally invest the same fraction in equities at all ages. Mossin (1968), Merton (1969) and Hakansson (1970) also have similar conclusions. Campbell and Viceira (2002) gave a detailed exposition on the conditions under which the investment horizon is irrelevant. Investors who have only financial wealth and who face constant investment opportunities should behave myopically, choosing the portfolio that has the best short-term characteristics. Booth and Yakoubov (2000) found no evidence in the postwar data to suggest that a lifestyle strategy is beneficial.

The key control variable in the deterministic lifestyle strategies, like in any asset allocation strategies, is the asset composition of the pension fund portfolio, which has a (higher risk asset to low risk asset) switch time and a simple asset composition-time (horizon) relationship. Blake et al (2001) have compared deterministic lifestyle and other two simple dynamic allocation strategies with two static ones by estimating their value-at-risk with Monte Carlo simulation. The two static strategies are a "50/50" allocation strategy with 50% in T-bills and 50% in bonds which was found to be the minimum-risk strategy for most asset-return models and a 'pension-fund-average' (PFA) strategy (Blake et al, 2001) which uses the average allocation of pension funds in UK and might be considered a high-risk strategy on account of its high equity weighting. The deterministic lifestyle has a

100% weighting in the PFA portfolio before a certain switching time t_s, followed by a $1/(T\text{-}t_s)$ per annum switch into the 50/50 portfolio during the remaining $T\text{-}t_s$ years of the accumulation phase. Blake et al (2001) found that a static asset allocation strategy with a high equity weighting (the 100% PFA strategy) delivers substantially better results than the lifestyle strategy and other two simple dynamic strategies.

The study of Blake et al (2001) does not examine whether the PFA allocation is optimal, and a high risk portfolio could be the optimal asset allocation. As at 31 March 2006, the PFA asset distribution is 35.8% in UK equities, 28.9% in overseas equities, 23.1% in bonds, 7.6% in index-linked gilts, 2.4% in property, 1.8% in cash, and 0.4% in other assets (Mellon Analytical Solutions: UK Pension Fund Analysis to 31 March 2006). The pension funds as a whole may follow the optimal asset allocation strategy for pension plans, but it is not certain that this is the case. Nor is it clear whether a 100% equity strategy will outperform the lifestyle strategies. In the present paper, I try to derive the optimal asset allocation strategy from a simpler two asset model and compare the optimal allocation as well as the 100% equity strategy with the deterministic lifestyle strategy.

Since the seminal studies of Samuelson (1969) and Merton (1969, 1971) on optimal consumption and portfolio strategies, optimal asset allocation problems have been solved under various assumptions (Kim and Omberg 1996; Brandt 1999; Sorensen 1999; Brennan et al 1997; Brennan and Xia 2000; Campbell and Viceira 1999, 2001; Barberis 2000; Liu 2001; Wachter 2002). The later studies demonstrate that time-varying investment opportunities result in an inter-temporal hedging component in the optimal portfolio composition, whose magnitude depends on the investment horizon. Even for constant investment opportunities, the optimal portfolio composition may be horizon-dependent under certain conditions. Samuelson (1989) found that if an individual needs to assure at retirement a minimum ("subsistence") level of wealth, she has greater risk-taking when young than old. Greater risk-taking may lead to a higher proportion of wealth invested in equities.

The existence of wage incomes also affects the optimal portfolio composition and its horizon dependence. Bodie et al (1992) show that investors endowed with a (non-tradable) riskless stream of labor income (human wealth) hold more risky assets in their youth than with only financial wealth. Campbell and Viceira (2002) suggest that in the model with constant relative risk aversion, labor income affects portfolio choice by reducing the proportional sensitivity of consumption to financial asset returns, thereby reducing the investor's aversion to financial risks. If labor income is riskless, a young, employed investor should invest more in stocks than a retired investor with identical risk aversion and financial wealth. This is consistent with the typical recommendation of financial advisors, but for different reasons. When labor income is not riskless, the impact of labor income on optimal portfolio proportions depends on the variance of labor income and the correlation between labor income and risky asset returns (Viceira 2001). A risky wage income with a positive correlation with stock returns tends to reduce the optimal allocation to risky assets compared with riskless labor incomes (and labor incomes negatively correlated with stock returns).

One important impact of wage income on the optimal portfolio problem is that analytical solution often cannot be derived for constant relative risk aversion (CRRA) utility with non-hedgeable wage risks. Most consumption and portfolio studies assume no wage income or fully hedgeable wage income so that analytical solution can be derived. Those studies dealing with non-hedgeable wage incomes either using numerical methods (Heaton and Lucas 1997; Koo 1998, 1999) or an approximate analytical solution (Campbell 1993; Campbell and Viceira 1999, 2001; Viceira 2001). In pension fund asset allocation strategy, the contribution from wage incomes has to be dealt with explicitly because of its prominent role in pension wealth growth. Deterministic or fully hedgeable wage incomes have been used in most pension fund strategy studies (Boulier et al 2001; Deelstra et al 2003; Vigna and Haberman 2001; Haberman and Vigna 2002; Cairns et al 2006). The optimal pension asset allocation strategy for CRRA utility with non-hedgeable wage risks has rarely been explored, nor is the comparison between lifestyle strategies and optimal pension asset allocation strategies with non-hedgeable wage risks.

The argument(s) in utility functions can change substantially the solution of optimal allocation problems, as shown by Samuelson (1989). The objective of pension funds is to maximize expected terminal utility. Boulier et al (2001) and Deelstra et al (2003) assume that the expected terminal utility is a function of lump sum cash over guaranteed minimum benefits, which can be considered as subsistence consumption. Cairns et al (2006) use replacement ratio or wealth-to-wage ratio, which take the current standard of living into account, as the argument of expected terminal utility. Taking current standard of living into account suggests a role of habit formation in the utility function (Spinnewyn 1981; Becker and Murphy 1988).

Defined-benefit pension plans are usually determined by final wages (and years of service), which also indicates that the need for post-retirement consumption has some influence from habit formation. In this paper, I assume that the expected terminal utility is a function of wealth-to-wage ratio. The use of wealth-to-wage ratio as the argument of the terminal utility function incorporates the wage risk and its correlation with the interest rate and stock returns into the optimal asset allocation decision. It also leads to a computational advantage that the optimal portfolio composition with commonly assumed stochastic interest rate, stock return, and wage income models (Battocchio and Menoncin 2004; Cairns et al 2006) is no longer horizon dependent.

Since dynamic allocation strategies tend to use two assets or two mutual funds and the switch is usually between the riskless asset/low-risk mutual and high-risk equities/mutual fund, I use two asset (either cash and equity or bond and equity) inter-temporal optimization models for the present investigation. In this paper, I first derive optimal portfolio composition for pension plans with fully hedgeable wage income and model the deterministic lifestyle allocation strategies mathematically; I then show that lifestyle allocation strategies can be replicated by a corresponding static allocation with same expected returns and less variance; I compare by numerical simulation the deterministic lifestyle allocation strategies with the (fully hedgeable wage) optimal allocation applied to fully hedgeable wage and non-hedgeable wage cases, which show that in both cases the optimal allocation outperforms the deterministic lifestyle strategy; finally I try to derive and compare optimal asset allocation for both fully hedgeable wage and non-hedgeable wage cases by numerical methods.

This paper is structured as follows. Section 2 presents the market model and derives the optimal asset allocation for two assets, cash (or bond) and stock. Section 3 derives a mathematic presentation of the deterministic lifestyle strategy and its static equivalent, which shows that the deterministic lifestyle strategy can be replicated by a static allocation with same expected returns and lower variances. Section 4 compares the lifestyle allocation strategies with the (fully hedgeable wage) optimal allocation applied to fully hedgeable wage and non-hedgeable wage cases and investigates by numerical methods the effects of values of non-hedgeable wage risk and pension contribution rate on the optimal asset allocation. Section 5 discusses and summarizes our results in this paper.

2. Model formulation and inter-temporal optimization

In this section I will present a market model with three types of asset, riskless assets, bonds and equities (stocks), and derive optimal asset allocation strategies for pension plans investing in riskless assets and equities, or bonds and equities. The reason to limit investment to two assets is for convenience in comparison with lifestyle strategies. Lifestyle strategies are most readily modelled with two assets.

2.1. Market structure and wealth growth model

The framework of financial market models in Boulier et al (2001), Deelstra et al (2003) and Battocchio and Menoncin (2004) are used here. The market is frictionless, and there is no transaction cost or constraint on short-sale. The uncertainty in the financial market is described by two standard and independent Brownian motions $Z_r(t)$ and $Z_S(t)$ with $t \in [0,T]$, defined on a complete probability space (Ω, $\mathcal{F}$, P) where P is the real world probability. The filtration $\mathcal{F} = \mathcal{F}$ (t) $\forall t \in [0,T]$ generated by the Brownian motions can be interpreted as the information set available to the investor at time t.

The instantaneous risk-free rate of interest $r(t)$ follows an Ornstein-Uhlenbeck process

$$dr(t) = \alpha(\beta - r(t))dt + \sigma_r dZ_r(t),$$

$$r(0) = r_0. \qquad (1)$$

In equation (1), α and β are strictly positive constants, and σ_r is the volatility of interest rate (Vasicek, 1977).

The price of zero-coupon bonds for any date of maturity τ at time t, $B(t, \tau, r)$, is governed by the diffusion equation (Vasicek 1977; Boulier et al 2001; Deelstra 2003)

$$\frac{dB(t,\tau,r)}{B(t,\tau,r)} = (r(t) + b(t,\tau)\sigma_r \xi)dt - b(t,\tau)\sigma_r dZ_r(t),$$

$$B(\tau,\tau) = 1,$$

where ξ is the market price of interest rate risk assumed to be constant, and

$$b(t,\tau) = \frac{1 - e^{-\alpha(\tau - t)}}{\alpha}.$$

There are three types of asset in the financial market: cash, bonds and equities. The riskless asset has a price process governed by

$$dR(t) = R(t)r(t)dt\,,$$
$$R(0) = R_0\,. \tag{2}$$

The riskless asset can be considered as a cash fund and the value of units in the cash fund at t is then

$$R(t) = R(0)\exp\left[\int_0^t r(s)ds\right]. \tag{3}$$

There are zero-coupon bonds for any date of maturity, and a bond rolling over zero coupon bonds with constant maturity K (Boulier et al, 2001). The price of the zero coupon bond with constant maturity K is denoted by $B_K(t, r)$ with

$$\frac{dB_K(t,r)}{B_K(t,r)} = [r(t) + b_K\sigma_r\xi]dt - b_K\sigma_r dZ_r(t) \quad , \tag{4}$$

where

$$b_K = \frac{1-e^{-\alpha K}}{\alpha} \quad .$$

For simplicity, only one equity asset, a stock, is considered, which can represent the index of a stock market. The total return (the value of a single premium investment in the stock with reinvestment of dividend income) of the stock follows the stochastic differential equation (SDE)

$$dS(t) = S(t)\left[\mu_S(r,t)dt + v_{rS}\sigma_r dZ_r(t) + \sigma_S dZ_S(t)\right],$$
$$S(0) = S_0\,, \tag{5}$$

where

$$\mu_S(r,t) = r(t) + m_S = r(t) + \sigma\xi_S = r(t) + \xi_S\sqrt{{v_{rS}}^2{\sigma_r}^2 + {\sigma_S}^2} \tag{6}$$

is the instantaneous percentage change in stock price per unit time. The total stock instantaneous volatility σ and the market price of stock risk ξ_S are assumed to be constant, and v_{rS} represents a volatility scale factor measuring how the interest rate volatility affects the stock volatility. The risk premium on the stock is $m_S = \sigma\xi_S$.

The market as assumed above has a diffusion matrix given by

$$\Sigma \equiv \begin{bmatrix} -b_K\sigma_r & 0 \\ v_{rS}\sigma_r & \sigma_S \end{bmatrix}, \tag{7}$$

and σ_r and σ_S are assumed to be different from zero and the diffusion matrix is invertible.

The plan member's wage, $Y(t)$, evolves according to the SDE (Battochio and Menoncin 2004; Cairns et al 2006)

$$dY(t) = Y(t)\left[(\mu_Y + r(t))dt + v_{rY}\sigma_r dZ_r(t) + v_{SY}\sigma_S dZ_S(t) + \sigma_Y dZ_Y(t)\right],$$

$$Y(0) = Y_0, \tag{8}$$

where $\mu_Y(t)$ and σ_Y are assumed to be constant for simplicity. Here $Z_Y(t)$ a standard Brownian motion independent of $Z_r(t)$ and $Z_S(t)$; v_{rY} and v_{SY} are volatility scaling factors measuring how interest rate volatility and stock volatility affect wage volatility, respectively. When $\sigma_Y = 0$, the market is complete. Otherwise the market is incomplete. The pension fund invests in stock and one of the two other assets, cash and bond. The reason for investigating both cash-stock and bond-stock strategies, is to take into account the difference between academic portfolio studies and fund management practices. Portfolio studies normally use cash (risk free) and stock (risky) in two assets models, while fund managers are more likely to use bond and stock.

When there is no non-hedgeable wage risk ($\sigma_Y = 0$), the fully hedgeable wage income is governed by

$$dY(t) = Y(t)[(r + \mu_Y)dt + v_{rY}\sigma_r dZ_r(t) + v_{SY}\sigma_S dZ_S(t)]. \tag{9}$$

If the pension fund invests in cash and stock, the SDE governing the wealth process is

$$\begin{aligned} dW(t) &= W(t)\left(\theta_R \frac{dR}{R} + \theta_S \frac{dS}{S}\right) + \pi Y(t)dt \\ &= [W(t)\theta_R r + W(t)\theta_S(r + m_S) + \pi Y(t)]dt + W(t)\theta_S v_{rS}\sigma_r dZ_r + W(t)\theta_S\sigma_S dZ_S. \end{aligned} \tag{10}$$

If the pension fund invests in bonds and stock, the SDE governing the wealth process is

$$\begin{aligned} dW(t) &= W(t)\left(\theta_B \frac{dR}{R} + \theta_S \frac{dS}{S}\right) + \pi Y(t)dt \\ &= [W(t)\theta_B(r + b_K\sigma_r\xi) + W(t)\theta_S(r + m_S) + \pi Y(t)]dt \\ &\quad + W(t)(-\theta_B b_K + \theta_S v_{rS})\sigma_r dZ_r + W(t)\theta_S\sigma_S dZ_S. \end{aligned} \tag{11}$$

2.2. Optimal asset allocation for power terminal utility

With fully hedgeable wage income, the market value at time t of future contributions payable between t and T is then

$$E_Q\left[\int_t^T \exp\left\{-\int_t^\tau r(s)ds\right\}\pi Y(\tau)d\tau \mid F_t\right]$$
$$=\pi E_Q\left[\int_t^T Y(t)\exp\left\{\int_t^\tau \mu_Y(s)ds-(\xi_r v_{rY}\sigma_r+\xi_S v_{SY}\sigma_S+\frac{1}{2}v_{rY}{}^2\sigma_r{}^2+\frac{1}{2}v_{SY}{}^2\sigma_S{}^2)(\tau-t)\right.\right.$$
$$\left.\left.+v_{rY}\sigma_r[\tilde{Z}_r(\tau)-\tilde{Z}_r(t)]+v_{SY}\sigma_S[\tilde{Z}_S(\tau)-\tilde{Z}_S(t)]\right\}d\tau \mid F_t\right]$$
$$=\pi Y(t)\int_t^T \exp\left\{\int_t^\tau \mu_Y(s)ds-(\xi_r v_{rY}\sigma_r+\xi_S v_{SY}\sigma_S)(\tau-t)\right\}d\tau$$
$$=\pi Y(t)f(t). \tag{12}$$

where Q is the risk-neutral pricing measure (Cairns et al 2006), ξ_r is a measure of how interest/bond volatility will affect wage, and ξ_S is a scale factor measuring how stock price volatility affects wages. The pension plan can have an additional wealth of $\pi Y(t)f(t)$ by short-selling a replicating portfolio of value $-\pi Y(t)f(t)$, which will be paid off exactly by future contributions from wage incomes. The total pension wealth enhanced with the present market value of future contributions is the augmented wealth $\tilde{W}(t)=W(t)+\pi Y(t)f(t)$.

Using Ito's lemma and substituting dW and dY, the process governing the augmented wealth-to-wage ratio $\tilde{X}(t)=\tilde{W}(t)/Y(t)$ can be written as

$$d\tilde{X}(t)=(\theta M+u)\tilde{X}dt+(\theta\Gamma'+\Lambda')\tilde{X}dZ, \tag{13}$$
$$\tilde{X}(T)=X(T).$$

Using θ as the proportion of wealth invested in stock, for pension funds investing in cash and stock,

$$M\equiv m_S-v_{rS}v_{rY}\sigma_r{}^2-v_{SY}\sigma_S{}^2,$$
$$u\equiv-\mu_Y+v_{rY}{}^2\sigma_r{}^2+v_{SY}{}^2\sigma_S{}^2,$$
$$\Gamma\equiv\begin{bmatrix}v_{rS}\sigma_r & \sigma_S\end{bmatrix}',$$
$$\Lambda\equiv\begin{bmatrix}-v_{rY}\sigma_r & -v_{sY}\sigma_S\end{bmatrix}',$$
$$Z\equiv\begin{bmatrix}Z_r & Z_S\end{bmatrix}'. \tag{14}$$

For pension funds investing in bond and stock, M, u and Γ are different

$$M\equiv m_S-b_K\sigma_r\xi-(v_{rY}b_K+v_{rS}v_{rY})\sigma_r{}^2-v_{SY}\sigma_S{}^2,$$
$$u\equiv b_K\sigma_r\xi-\mu_Y+v_{rY}(v_{rY}+b_K)\sigma_r{}^2+v_{SY}{}^2\sigma_S{}^2,$$
$$\Gamma\equiv\begin{bmatrix}(b_K+v_{rS})\sigma_r & \sigma_S\end{bmatrix}'. \tag{15}$$

The stochastic optimal control problem for terminal utility that is a function of terminal wealth-to-wage ratio is:

$$\max_{\theta} E\left[U(X(T),T)\right]$$

subject to

$$d\begin{bmatrix} w \\ \tilde{X} \end{bmatrix} = \begin{bmatrix} \mu_w \\ \theta M\tilde{X} \end{bmatrix} dt + \begin{bmatrix} \Omega' \\ \theta\Gamma'\tilde{X} \end{bmatrix} dZ ,$$

$$w(0) = w_0, \tilde{X}(0) = \tilde{X}_0, \forall 0 \le t \le T , \qquad (16)$$

where,

$$\underset{2\times1}{w} \equiv \begin{bmatrix} r & Y \end{bmatrix}',$$

$$\underset{2\times1}{\mu_w} \equiv \begin{bmatrix} \alpha(\beta - r) & Y(\mu_Y + r) \end{bmatrix}', \qquad (17)$$

$$\underset{2\times2}{\Omega'} \equiv \begin{bmatrix} \sigma_r & 0 \\ Yv_{rY}\sigma_r & Yv_{SY}\sigma_S \end{bmatrix}.$$

The corresponding Hamilton-Jacobi-Bellman equation is

$$H(J) = J_t + \mu'_w \frac{\partial J}{\partial w} + (\theta M + u)\tilde{X}\frac{\partial J}{\partial \tilde{X}} + \frac{1}{2} tr\left(\Omega'\Omega \frac{\partial^2 J}{\partial w^2}\right) + (\theta\Gamma' + \Lambda')\Omega\tilde{X}\frac{\partial^2 J}{\partial w \partial \tilde{X}}$$

$$+ \frac{1}{2}(\theta\Gamma'\Gamma\theta + 2\theta\Gamma'\Lambda + \Lambda'\Lambda)\tilde{X}^2 \frac{\partial^2 J}{\partial \tilde{X}^2}. \qquad (18)$$

The system of the first order conditions on H with respect to θ is

$$\frac{\partial H}{\partial \theta} = M\tilde{X}\frac{\partial J}{\partial \tilde{X}} + \Gamma'\Omega \frac{\partial^2 J}{\partial w \partial \tilde{X}}\tilde{X} + (\Gamma'\Gamma\theta + \Gamma'\Lambda)\tilde{X}^2 \frac{\partial^2 J}{\partial \tilde{X}^2} = 0 . \qquad (19)$$

The optimal portfolio composition is

$$\theta^* = -(\Gamma'\Gamma)^{-1}\Gamma'\Lambda - (\Gamma'\Gamma)^{-1} M \frac{J_{\tilde{X}}}{\tilde{X}J_{\tilde{X}\tilde{X}}} - (\Gamma'\Gamma)^{-1}\Gamma'\Omega \frac{J_{w\tilde{X}}}{\tilde{X}J_{\tilde{X}\tilde{X}}} . \qquad (20)$$

Equation (20) shows that the optimal allocation in the stock contains three components, which is consistent with earlier studies (Battocchio and Menoncin 2004; Cairns et al 2006).

Assuming that the maximized expected terminal utility of plan members has the functional form

$$J(t,x,w) = \frac{1}{1-\gamma} g(t,w)^{\gamma} x^{1-\gamma} , \qquad (21)$$

the optimal asset allocation problem can be solved analytically. The optimal proportion invested in stocks for pension funds investing in cash and stock is

$$\theta^* = -(\Gamma'\Gamma)^{-1}\Gamma'\Lambda + (\Gamma'\Gamma)^{-1} M \frac{1}{\gamma}$$
$$= \frac{v_{rS} v_{rY} \sigma_r^{\ 2} + v_{SY}\sigma_S^{\ 2}}{v_{rS}^{\ 2}\sigma_r^{\ 2} + \sigma_S^{\ 2}} + \frac{m_S - v_{rS} v_{rY} \sigma_r^{\ 2} - v_{SY}\sigma_S^{\ 2}}{(v_{rS}^{\ 2}\sigma_r^{\ 2} + \sigma_S^{\ 2})\gamma}. \qquad (22)$$

The optimal proportion of pension wealth invested in stocks for pension funds investing in bond and stock is

$$\theta^* = \frac{(b_K + v_{rY})(b_K + v_{rS})\sigma_r^{\ 2} + v_{SY}\sigma_S^{\ 2}}{(b_K + v_{rS})^2\sigma_r^{\ 2} + \sigma_S^{\ 2}} + \frac{m_S - b_K\sigma_r\xi - (b_K + v_{rS})v_{rY}\sigma_r^{\ 2} - v_{SY}\sigma_S^{\ 2}}{\gamma[(b_K + v_{rS})^2\sigma_r^{\ 2} + \sigma_S^{\ 2}]} \qquad (23)$$

The optimal proportion of pension wealth invested in cash or bonds is $1-\theta^*$. The optimal portfolio composition is horizon independent. For details of the above derivation see Appendix A. From the above analysis, we have the following proposition:

Proposition 1: *For a pension plan of individuals with fully hedgeable wage income and constant relative risk aversion (CRRA) utility, the optimal portfolio composition is horizon independent*.

Proposition 1 indicates that for fully hedgeable wage incomes, a deterministic lifestyle strategy is unlikely to be optimal.

3. Deterministic lifestyle

In order to compare with inter-temporal optimization, I start the analysis with a pure equity (stock) strategy as the high risk strategy instead of using the PFA. The deterministic lifestyle strategy is assumed to be shifting from equities to riskless assets or/and low risk bonds over the lifetime of the pension plan. The “optimization” problem in a deterministic life style strategy can be characterized in the following way:

Let t(0)=0 be the pension plan starting date, T be the pension fund mature date. In the market structure as described above, choose an optimal switching-starting time t_s, $0 \le t_s < T$. All pension wealth will be invested in equities before t_s, and after t_s, a proportion

$$\theta = \frac{t - t_s}{T - t_s} \quad (24)$$

of the pension wealth will be invested in the low risk cash fund or bonds. Two different lifestyle strategies are considered: one switches from equities to cash fund (equity-cash), and the other switches from equities to bonds (equity-bond). Before I try to show that there is no unique optimal switching time for lifestyle strategies, I will first demonstrate that the simple deterministic lifestyle strategy can be replicated by a static allocation with same expected returns and smaller variances.

3.1. Replicating deterministic lifestyle strategy with a static allocation

Markowitz's pioneering work on mean-variance analysis demonstrates that portfolios with same expected returns can have vary different variance, the risk averse investors should choose the one with least variance (Markowitz 1952). Samuelson (1969) showed that a rational maximizer of expected utility, with constant relative risk aversion and facing random-walk securities returns, would rationally invest the same fraction in equities at all ages rather than varying the proportions at different ages. Mark Kritzman (2000) more illustratively pointed out in a chapter entitled "Half stocks all the time or all stocks half the time?" in his book *Puzzles of Finance,* that these two strategies have the same expected simple return, but the later is riskier. It is easy to show that simple lifestyle strategies can be better replicated with static allocation strategies, and the corresponding static strategy has the same expected return and is less risky.

For simplicity, the interest rate is assumed to be constant here so that cash and bond assets are the same. Other assumptions are kept the same as those in section 2; the initial wealth is W_0 and there is no further contribution. Let the switching time be t_s. The wealth process before the switching time is simply

$$\frac{dW(t)}{W(t)} = (r + m_S)dt + \sigma_S dZ_S(t) .$$

And the wealth process after the switching time is

$$\frac{dW(t)}{W(t)} = \frac{t - t_S}{T - t_S} rdt + \frac{T - t}{T - t_S}\left[(r + m_S)dt + \sigma_S dZ_S(t)\right].$$

The terminal value of the first process is the initial value of the second process. The expected terminal pension wealth at the beginning of the pension plan should be the expected terminal value of the second process.

The expected terminal pension wealth is

$$E_{t_s}\left[W(T)\right] = W(t_s)E_{t_s}\left\{\exp\left(\int_{t_s}^{T}\frac{t-t_s}{T-t_s}rdt + \int_{t_s}^{T}\frac{T-t}{T-t_s}\left[(r+m_S)dt + \sigma_S dZ_S(t)\right]\right)\right\}$$
$$= W(t_s)E_{t_s}\left\{\exp\left[\frac{r}{2}(T-t_s) + \frac{1}{2}(T-t_s)(r+m_S) + \int_{t_s}^{T}\frac{T-t}{T-t_s}\sigma_S dZ_S(t)\right]\right\}$$
$$= W(t_s)\exp\left[\frac{r}{2}(T-t_s) + \frac{1}{2}(T-t_s)(r+m_S)\right].$$

It is easy to see that the gradual switch between riskless and risky assets has the same expected terminal wealth as a static 50/50 allocation. The expected t_s value of pension wealth at the beginning of pension plan is

$$E_0\left[W(t_s)\right] = W(0)E_{t_0}\left\{\exp\int_0^{t_s}\left[(r+m_S)dt + \sigma_S dZ_S(t)\right]\right\}$$
$$= W(0)E_0\left\{\exp\left[t_s(r+m_S) + \sigma_S Z_S(t_s)\right]\right\}$$
$$= W(0)\exp\left[t_s(r+m_S)\right].$$

Combining the two results, we have

$$E_0\left[W(T)\right] = W(0)\exp\left[t_s(r+m_S)\right]\exp\left[\frac{r}{2}(T-t_s) + \frac{1}{2}(T-t_s)(r+m_S)\right]$$
$$= W(0)\exp\left[\frac{r}{2}(T-t_s) + \frac{1}{2}(T-t_s)(r+m_S) + t_s(r+m_S)\right] \qquad (25)$$
$$= W(0)\exp\left\{rT\left[\frac{T-t_s}{2T}\right] + \left[(r+m_S)T\left(\frac{T+t_s}{2T}\right)\right]\right\}.$$

The above equation shows that a simple lifestyle strategy with a switching time t_s can be readily replicated in terms of expected terminal wealth by a static allocation portfolio with a proportion of $\frac{T-t_s}{2T}$ invested in the riskless asset and a proportion of $\frac{T+t_s}{2T}$ invested in the risky asset.

While the annualized variance of the lifestyle portfolio return will vary along with the changing proportion invested in the risky asset, the annualized variance for the static allocation portfolio is constant. Fig. 1 uses parameters given in Table 1 and illustrates the terminal wealth distributions from 3 different switching times, from beginning (t_s=0), t_s=15 and t_s=30 for a plan of 45 years and their corresponding static replicating schemes. The values in Table 1 are commonly used in other pension studies (Boulier et al 2001; Deelstra et al 2003; Cairns et al 2006; Battocchio and Menoncin 2004) and chosen to facilitate comparison with those earlier studies. The difference in the cumulative wealth distribution density curve between any of the lifestyle schemes and their corresponding static ones is small, but the static allocations have a smaller probability for lower wealth.

Table 1 Parameters used in numerical simulation

Interest rate	Main value	Alternative value
Mean reversion, α,	0.2	
Mean rat, β	0.05	
Volatility, σ_r	0.02	
Initial rate, r_0	0.05	
Fixed maturity bond		
Maturity, K	20 years	
Market price of risk, ξ	0.15	
Stock		
Risk Premium, m_S	0.06	
Stock own volatility, σ_S	0.19	
Interest volatility scale factor, v_{rS}	1	
Wage		
Wage premium, μ_Y	0.01	
Non-hedgeable volatility, σ_Y	0.01	
Interest volatility scale factor, v_{rY}	0.7	
Stock volatility scale factor, v_{SY}	0.9	
Initial wage, Y_0	10k	
Contribution rate, π	10%	
Risk aversion		
Relative risk aversion, γ	2	0.8
Length of pension plan, T	45	

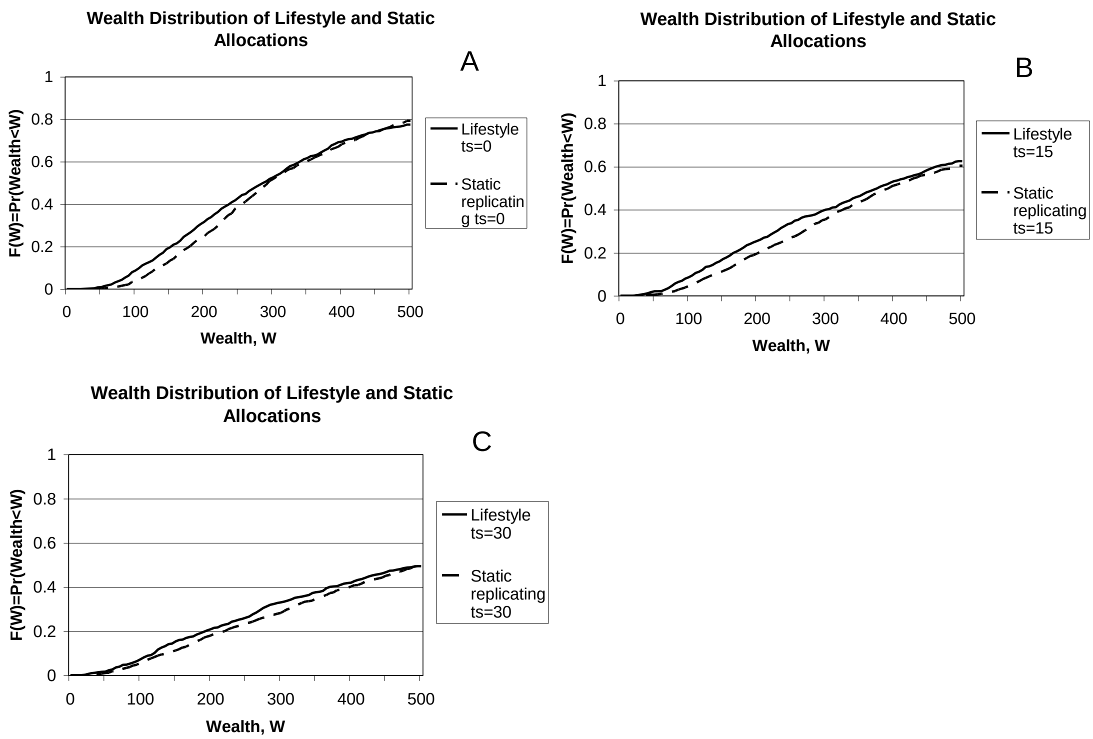

Fig.1 Distribution of terminal pension wealth (wealth at retirement) of lifestyle strategies and their corresponding static strategies. Corresponding static schemes are 50% in risk free assets and 50% in equities for *ts*=0, 1/3 in risk free assets and 2/3 in equities for *ts*=15, and 1/6 in risk free assets and 5/6 in equities for *ts*=30. Results are from 1000 simulations of wealth growth paths.

To compare the variances of the lifestyle strategy and its corresponding static allocation, the logarithms of portfolio values are used because they are computationally simpler. The wealth process before the switching time is simply

$$d(\ln W(t)) = \frac{dW(t)}{W(t)} = (r + m_S)dt + \sigma_S dZ_S(t). \tag{26}$$

The variance of $(\ln W(t_s) - \ln W(0))$ is

$$\mathrm{var}(\ln W(t_s) - \ln W(0)) = \sigma_S{}^2 t_s .$$

And the wealth process after the switching time is

$$d(\ln W(t)) = \frac{dW(t)}{W(t)} = \frac{t - t_S}{T - t_S} r dt + \frac{T - t}{T - t_S}\left[(r + m_S)dt + \sigma_S dZ_S(t)\right].$$

The variance of $(\ln W(T) - \ln W(t_s))$ is

$$\mathrm{var}(\ln W(T)-\ln W(t_s)) = E\left(\int_{t_s}^{T}\frac{T-t}{T-t_s}\sigma_S dZ_S(t)\right)^2 = E\int_{t_s}^{T}\left(\frac{T-t}{T-t_s}\sigma_S\right)^2 dt$$
$$=\frac{T-t_s}{3}\sigma_S{}^2 .$$

The Itô isometry $E\left(\int_S^T\phi(t,\omega)dZ(t,\omega)\right)^2 = E\int_S^T(\phi(t,\omega))^2 dt$ is used in the second equality (Øksendal 2000, p26). Since $\mathrm{var}(\ln W(0))=0$,

$$\begin{aligned}&\mathrm{var}(\ln W(T)) = \mathrm{var}(\ln W(T)-\ln W(t_s)) + \mathrm{var}(\ln W(t_s)-\ln W(0))\\&=\frac{T-t_s}{3}\sigma_S{}^2+\sigma_S{}^2 t_S = \frac{T+2t_s}{3}\sigma_S{}^2 .\end{aligned} \tag{27}$$

The corresponding static allocation process is

$$d(\ln W(t)) = \frac{dW(t)}{W(t)} = \frac{T-t_S}{2T}rdt + \frac{T+t_s}{2T}\left[(r+m_S)dt+\sigma_S dZ_S(t)\right]. \tag{28}$$

The variance of $\mathrm{var}(\ln W(T))$ is

$$\mathrm{var}(\ln W(t)) = \frac{(T+t_s)^2}{4T}\sigma_S{}^2 \quad . \tag{29}$$

The difference between lifestyle and corresponding static allocation is

$$\frac{T+2t_s}{3}\sigma_S{}^2 - \frac{(T+t_s)^2}{4T}\sigma_S{}^2 = \frac{T^2+2Tt_s-3t_s{}^2}{12T}\sigma_S{}^2 \geq 0, \qquad \forall t_s \leq T^2$$

$$\mathrm{var}(lifestyle) \geq \mathrm{var}(static), \qquad \forall t_s \leq T$$

Therefore, the replicating static portfolio has the same expected return and a smaller variance, i.e., the replicating static allocation second-order stochastically dominates the lifestyle strategy.

In Fig.1, the difference in variances between the lifestyle schemes and the corresponding static allocations appears to be small. The maximum additional variance can be derived from the first-order condition of the function $f(t_s)=\frac{T^2+2Tt_s-3t_s{}^2}{12T}\sigma_S{}^2$, which is $\frac{T}{9}\sigma_S{}^2$ when $t_s{*}=\frac{T}{3}$. This result is supported by the *ts*=15 (T/3, T=45) curves in Fig.1B, they appear to have the largest difference among the three groups. The maximum relative difference in their variances can be derived by the first order condition of the function

$$g(t_s)=\frac{\frac{T^2+2Tt_s-3t_s{}^2}{12T}\sigma_S{}^2}{\frac{(T+t_s)^2}{4T}\sigma_S{}^2} .$$

Differentiating $g(t_s)$ with respect to t_s yields $t_s{}^* = 0$ and the maximum relative difference 1/3, or approximately 33.3%. Given $\sigma_S = 0.19$ and the wealth involved in the simulation, a 1/3 increase in the variance may not be obvious. This may partly explain why the lifestyle strategy is still popular with many financial advisors: the actual difference is probably not big enough to show a definite advantage of the static allocation. Based on these results, we have

Proposition 2: *The simple deterministic lifestyle asset allocation strategy can be replicated by a static allocation strategy with same expected return and smaller variance (when the instantaneous interest rate is constant). The static allocation strategy second order dominates its corresponding lifestyle strategy*.

The preceding analysis also applies to scenarios with stochastic short interest rates and risk premiums. To derive the replicating proportions invested in the two assets in the case of stochastic short interest rates and risk premiums, explicit function forms of $r(t)$ and $m_S(t)$ need to be specified.

In this subsection I have shown that the simple deterministic lifestyle strategy can be readily replicated with a static allocation strategy with less risk. In the following subsections I am going to investigate whether there is an optimal switching time for lifestyle strategies.

3.2. Lifestyle strategy for power terminal utility

In section 2 I have derived the optimal asset allocation for two assets and the optimal proportion of stock is constant in both cases. The lifestyle strategy is in fact equivalent to providing an allocation strategy for HJB equation with proportion of stock

$$\theta(t_s, t) \equiv \min\left(1, \frac{T-t}{T-t_s}\right). \tag{30}$$

The optimal switching starting time t_s (i.e. strategy t_s) will maximize terminal utility if such an optimal switching time exists.

Obviously there is no such t_s that can make the min function, $\min\left(1, \frac{T-t}{T-t_s}\right)$, satisfy a constant optimal proportion from intertemporal optimization. Therefore, there is no such optimal switching time in a deterministic lifestyle strategy that can optimize asset allocation for an individual with power terminal utility function.

I conjecture that the best or "optimal" switching time for a deterministic lifestyle strategy is such a time point that the average proportion over the accumulation phase invested in equities equals to the above optimal condition for θ from inter-temporal optimization. The average proportion over accumulation phase for a deterministic lifestyle strategy can be calculated with

$$\bar{\theta} = \frac{t_s}{T} + \frac{(T - t_s)}{2T} = \frac{T + t_s}{2T}.$$

This is to find the lifestyle equivalent of a static allocation with the same expected return, a reverse of what was done earlier – to find the static equivalent of a lifestyle strategy. Using this equation I hypothesize that the best switching time for an equity-cash lifestyle strategy is

$$t_s = 2T\left(\frac{v_{rS}v_{rY}\sigma_r{}^2 + v_{SY}\sigma_S{}^2}{v_{rS}{}^2\sigma_r{}^2 + \sigma_S{}^2} + \frac{m_S - v_{rS}v_{rY}\sigma_r{}^2 - v_{SY}\sigma_S{}^2}{(v_{rS}{}^2\sigma_r{}^2 + \sigma_S{}^2)\gamma}\right) - T. \qquad (31)$$

Whether there is a $t_s < T$ depends on the first term in the brackets on the right-hand-side. If the first term is less than 1, which implies $(v_{rS}v_{rY} - v_{rS}{}^2)\sigma_r{}^2 < (1 - v_{SY})\sigma_S{}^2$, there must be a $t_s < T$ when γ approaches infinity. This condition is satisfied with usual (or other plausible) market parameters.

The best switching time for an equity-bond lifestyle strategy is

$$t_s = 2T\left(\frac{(b_K + v_{rY})(b_K + v_{rS})\sigma_r{}^2 + v_{SY}\sigma_S{}^2}{(b_K + v_{rS})^2\sigma_r{}^2 + \sigma_S{}^2} + \frac{m_S - b_K\sigma_r\xi - (b_K + v_{rS})v_{rY}\sigma_r{}^2 - v_{SY}\sigma_S{}^2}{\gamma[(b_K + v_{rS})^2\sigma_r{}^2 + \sigma_S{}^2]}\right) - T.$$

(32)

The condition for $t_s < T$ when γ approaches infinity is $[(v_{rY} - v_{rS})b_K + v_{rS}v_{rY} - v_{rS}{}^2]\sigma_r{}^2 < (1 - v_{SY})\sigma_S{}^2$, which is also satisfied with usual (or other plausible) market parameters. These switching times will produce the same expected terminal wealth as that of inter-temporal optimization, but with a larger variance. From the above equations, an "implied relative risk aversion" can be calculated if a preferred switching time ratio t_s / T in the equity-cash strategy exists for an individual,

$$\gamma = \frac{m_S - v_{rS} v_{rY} \sigma_r^2 - v_{SY} \sigma_S^2}{\frac{1}{2}(v_{rS}^2 \sigma_r^2 + \sigma_S^2)\left(\frac{t_s}{T} + 1\right) - v_{rS} v_{rY} \sigma_r^2 - v_{SY} \sigma_S^2}. \tag{33}$$

The "implied relative risk aversion" if a preferred switching time ratio t_s / T in the equity-bond strategy exists for an individual is,

$$\gamma = \frac{m_S - b_K \sigma_r \xi - (b_K + v_{rS}) v_{rY} \sigma_r^2 - v_{SY} \sigma_S^2}{\frac{1}{2}[(b_K + v_{rS})^2 \sigma_r^2 + \sigma_S^2]\left(\frac{t_s}{T} + 1\right) - (b_K + v_{rY})(b_K + v_{rS}) \sigma_r^2 - v_{SY} \sigma_S^2}. \tag{34}$$

The above relations between the "implied relative risk aversion" and switching time ratio t_s / T are drawn in Fig. 2. With the parameters in Table 1, starting from a 100% stock strategy the "implied relative risk aversion" increases as switching time ratio t_s / T becomes smaller (switch earlier). The "implied relative risk aversion" in a equity-cash strategy goes to infinity as t_s / T approaches the point where

$$\frac{t_s}{T} = \frac{2(v_{rS} v_{rY} \sigma_r^2 + v_{SY} \sigma_S^2)}{v_{rS}^2 \sigma_r^2 + \sigma_S^2} - 1.$$

This result is consistent with the fact that switching is usually started in the later years of a pension plan. As shown in Fig.2A, a switching time ratio of 0.8 (equivalent to 8 years before retirement for 40 years plan) implies a relative risk aversion (RRA) well above 100 (the value is 340).

The point where the "implied relative risk aversion" in an equity-bond strategy goes to infinity is

$$\frac{t_s}{T} = \frac{2[(b_K + v_{rS})(b_K + v_{rY}) \sigma_r^2 + v_{SY} \sigma_S^2]}{(b_K + v_{rS})^2 \sigma_r^2 + \sigma_S^2} - 1.$$

As shown in Fig. 2B with the bond-stock portfolio, a switching time ratio of 0.83 implies a RRA well above 60 (the value is 169). When the switching time ratio is greater than 0.9, the "implied relative risk aversion" is in a plausible range of 2.584-6.142. The value of 2.584 corresponds to $t_s / T = 1$.

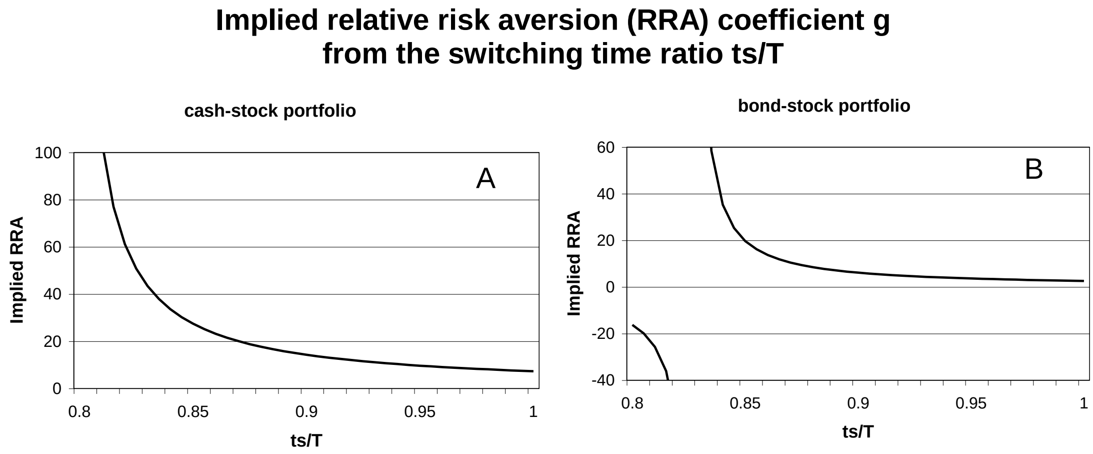

Fig. 2 The relationship between the switching time ratio t_s / T of lifestyle strategy and its "implied relative risk aversion". A. In cash-stock portfolios. B. In bond-stock portfolios.

4. Comparison by numerical methods

In this section, I compare the expected terminal utility between lifestyle strategies and the optimal asset allocation strategy by numerical simulation. I also try to derive the optimal asset allocation for non-hedgeable risk by numerical methods.

4.1. Parameters for numerical simulation

Different values of v_{rS} (for example, -1 and 1) could lead to very different optimal bond and cash proportions for more risk averse investors (larger γ) (Ma 2007). I have tested both $v_{rS} = -1$ and $v_{rS} = 1$. Since there is no substantial difference in the optimal allocations between $v_{rS} = -1$ and $v_{rS} = 1$ with the two γ values (0.8 and 2) in Table 1, I will only present the results from the $v_{rS} = 1$ case. Table 2 shows the optimal proportions invested in different assets for different allocation strategies with parameters in Table 1.

In Table 2, the proportions are analytical solution for fully hedgeable wage income ($\pi \neq 0$ and $\sigma_Y = 0$), but they have also been used for testing numerically the scenario where $\pi \neq 0$ and $\sigma_Y \neq 0$. These tests show that the presence of non-hedgeable risk has only a small effect on the performance of different portfolios and the optimal asset allocation, suggesting that if the optimal portfolio composition for $\pi \neq 0$ and $\sigma_Y \neq 0$ scenario is solved numerically, it would be similar to the optimal composition for $\pi \neq 0$ and $\sigma_Y = 0$ scenario. Minus sign (-) indicates short-sale; -0.271 in cash means short-selling cash asset valued as 27.1% of the net pension wealth. With short-sale of cash asset, the proportion invested in stock is 127.1% of the net pension wealth. For the power utility, the optimal proportions are dependent on the relative risk aversion.

Table 2 Optimal proportions invested in different assets

Strategies	Utility	Cash	Bond	Stock
Cash-stock	power	-0.271		1.271
Bond-stock	power		-0.0252	1.0252

4.2. Numerical simulation method

The Euler-Maruyama method is used for numerical simulation of stochastic differential equation (Higham 2001). The SDE

$$dX(t) = f(X(t))dt + g(X(t))dZ(t) \tag{35}$$

is simulated over [0,T] by using

$$X_j = X_{j-1} + f(X_{j-1})\Delta t + g(X_{j-1})(Z(\tau_j) - Z(\tau_{j-1}) \quad \text{j=1,2,…,N.} \tag{36}$$

In the above difference equation, $\Delta t = T/N = Rh$, $\tau_j = j\Delta t$ and

$$Z(\tau_j) - Z(\tau_{j-1}) = Z(jRh) - Z((j-1)Rh) = \sum_{k=jR-R+1}^{jR} dZ_k \,. \tag{37}$$

The terminal utility is calculated from the terminal wealth-to-wage ratio of each simulation and 1000 simulations performed for each allocation strategy. The cumulative terminal utility distribution density as well as the mean and the standard deviation (SD) are then calculated for each allocation strategy.

4.3. Comparison between lifestyle strategies and inter-temporal optimization

Three lifestyle strategies, two inter-temporal optimal strategies, 100% cash or 100% bond, and 100% equity (stock) strategies are compared by numerical simulation using parameters in Tables 1 and 2. In the lifestyle strategies, the wage contribution is added to the pension wealth as it comes in and no short-sale involved. In the inter-temporal optimal allocation strategy “optimal power, augmented wealth” (“power solution borrow” in Fig.3), the present value of future wage contributions is used as pension wealth by short-selling a wage replicating portfolio, which will be paid off by future wage contributions. In the inter-temporal optimal allocation strategy “optimal power, non-augmented” (“power solution no borrow” in Fig.3), the wage contribution is added to the pension wealth as it comes in and no short-sale of the wage replicating portfolio is involved. In the 100% cash, 100% bond, and 100% equity (stock) strategies, the wage contribution is added to the pension wealth as it comes in and no short-sale of the wage replicating portfolio is involved.

Numerical simulations on the expected utility and the utility distribution with parameters given in Tables 1 and 2 indicate that optimization for power terminal utility dominates the lifestyle strategy with three different switching times for both the equity-cash and the equity-bond cases (Tables 3 and Fig.3). Even when the optimal allocation derived from borrowing against future wages (“Optimal power, augmented wealth”) is used for allocating contribution from wage income and cumulated wealth only (without transforming the future wage income contributions into an initial augmented pension wealth by short-selling a replicating portfolio) (“Optimal power, non-augmented”), the inter-temporal optimization still has a higher expected terminal utility than the lifestyle strategy. In the “Optimal power, non-augmented” case, short-sale in implementing the optimal asset allocation derived for $\sigma_Y = 0$ scenario is still allowed; “non-augmented” means no short-selling the replicating portfolio in order to transform the future wage income contributions into the initial augmented pension wealth. The 100% stock strategy with parameters commonly used in pension studies also has a higher expected terminal utility than the lifestyle strategies. In contrast, the 100% cash or 100% bonds strategy is least efficient (Table 3). These results are consistent with the findings by Blake et al (2001).

When the optimal asset allocation derived for $\sigma_Y = 0$ scenario is applied for the $\pi \neq 0$ and $\sigma_Y \neq 0$ scenario, both the “optimal power, augmented wealth” and “optimal power, non-augmented” cases still dominate the lifestyle strategies. There is little difference in the performance of the optimal asset allocation between the $\sigma_Y = 0$ scenario and the $\pi \neq 0$ and $\sigma_Y \neq 0$ scenario (Table 3 and Fig.3). The terminal utility of all asset allocation strategies in the $\sigma_Y \neq 0$ scenario is lower than that in the $\sigma_Y = 0$ scenario for power utility (Tables 3), suggesting that an extra risk source reduces utility.

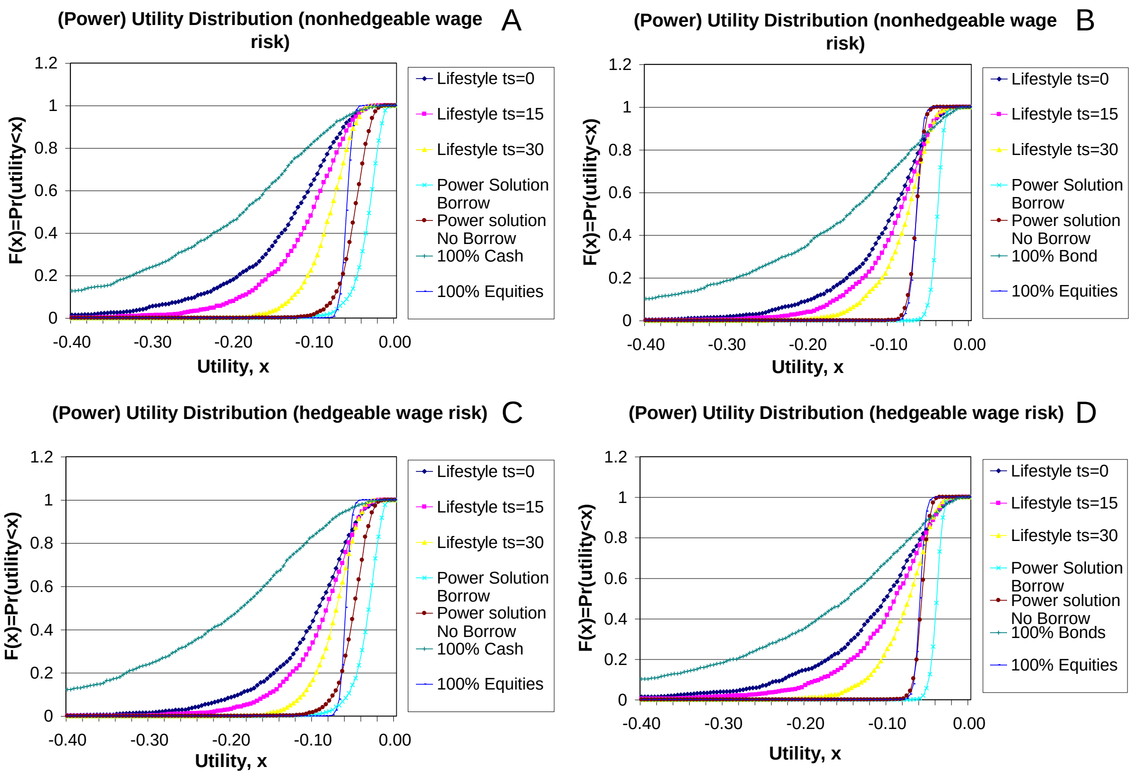

Fig.3 Comparison between simple deterministic lifestyle strategies and intertemporal optimization, evaluated for power utility, γ=2. Results are from 1000 simulations. "Power solution borrow" (i.e. with augmented wealth) is to invest in optimal proportions the cash borrowed by hedging against future wage income. "Power solution no borrow" (i.e. with non-augmented wealth) is to invest the contemporary wage income and cumulated wealth according to the optimal proportions without borrowing cash against future wage income. A. Equity-cash strategy, $\sigma_Y \neq 0$. B. Equity-bond strategy, $\sigma_Y \neq 0$. C. Equity-cash strategy, $\sigma_Y = 0$. D. Equity-bond strategy, $\sigma_Y = 0$

The order of expected terminal utility for different allocation strategies using bonds and stock is the same as that using cash and stock. The 100% bonds strategy has larger expected terminal utility than 100% cash strategy (Tables 3). When γ=0.8 is used in the simulations, the change in relative risk aversion does not change the order of expected terminal utility for different allocation strategies. From those numerical results, we have the following proposition.

Proposition 3: *The optimal asset allocation from intertemporal optimization, applied to both hedgeable and non-hedgeable wage cases, outperforms the simple lifestyle asset allocation strategy when the expected terminal utility is a function of wealth-to-wage ratio. With the current market parameters, the high risk 100% equity strategy also outperforms the simple lifestyle strategy*.

Table 3 Expected utility of lifestyle strategies and optimal asset allocation

Strategy	$\sigma_Y = 0$			$\sigma_Y \neq 0$	
	Equity-cash, γ=2	Equity-cash, ,γ=0.8	Equity-bond, γ=2	Equity-cash, , γ=2	Equity-bond, γ=2
t_s=0	-0.10816	8.085152	-0.12331	-0.14206	-0.12717
t_s=15	-0.09313	8.260456	-0.10353	-0.11486	-0.10857
t_s=30	-0.07580	8.479618	-0.08073	-0.08498	-0.08656
Optimal power, augmented wealth	-0.03516	10.547960	-0.04023	-0.03534	-0.04031
Optimal power, non-augmented	-0.05176	9.714139	-0.05809	-0.05179	-0.05809
100% cash	-0.22877	7.069287		-0.22905	
100% bond			-0.19374		-0.19393
100% equity	-0.06009	8.783799	-0.06009	-0.06014	-0.06014

The results shown in Fig.3 and Table 3 are from simulations with RRA=2, which is at the lower end of usual RRA estimates. Simulations with RRA=6 produced similar results, although the optimal proportion invested in stocks is lower for larger RRA value. The optimal proportions with RRA=6 and other parameters in Table 1 are shown in Table 4. The short-sale of cash assets in the cash-equity scenario is smaller than that for RRA=2, and there is a positive holding of bonds in the bond-equity scenario (whereas there is a short-sale of bond for RRA=2).

Table 4 Optimal proportions invested in different assets with RRA=6

Strategies	Utility	Cash	Bond	Stock
Cash-stock	power	-0.02215		1.022146
Bond-stock	power		0.049143	0.950857

Table 5 Expected utility of lifestyle strategies and optimal asset allocation ($\times 10^{-4}$) with RRA=6

Strategy	$\sigma_Y = 0$		$\sigma_Y \neq 0$	
	Equity-cash, γ=6	Equity-bond, γ=6	Equity-cash, , γ=6	Equity-bond, γ=6
t_s=0	-0.43051	-1.83301	-1.48325	-0.49921
t_s=15	-0.11739	-0.42752	-0.32436	-0.14434
t_s=30	-0.01635	-0.0346	-0.02924	-0.03799
Optimal power, augmented wealth	-0.00003	-0.00024	-0.00027	-0.00025
Optimal power, non-augmented	-0.00164	-0.00155	-0.00168	-0.00158
100% cash	-25.965		-26.3845	
100% bond		-30.8723		-31.9074
100% equity	-0.00171	-0.00171	-0.00176	-0.00176

Table 5 summarizes the expected utility of different allocation strategies for RRA=6 by numerical simulation. The general pattern is similar to that of RRA=2, with the inter-temporal optimal allocation with augmented wealth producing the best outcome and without short-selling wage replicating portfolio (non-augmented) producing the second best outcome. The 100% equity strategy leads to better outcomes than the lifestyle strategies. The 100% cash and 100% bond strategies have the worst outcomes.

The results of numerical simulation summarized in Fig.3 and Tables 3 and 5 indicate that the inter-temporal optimal allocation strategy dominates the deterministic lifestyle strategies with the usual assumptions on market parameters and the usual estimates on relative risk aversion coefficient, no matter whether it is applied by short-selling a wage replicating portfolio or just adding the wage contribution to pension wealth as it comes in. The high risk 100% equity strategy also outperforms the deterministic lifestyle strategies.

4.4. Simulation with lower equity risk premium

In the numerical simulation of preceding subsection, I have used an equity risk premium of 0.06 (Table 1), which is an estimate based on historical stock return data in the United States. It has been argued that average stock returns are likely to be lower in the future than they have been in the past (Blanchard 1993; Campbell and Shiller 2001; Fama and French 2002; Jagannathan, McGrattan and Scherbina 2001). Using an equity risk premium of 0.04 is a fairly common choice in recent literature (Fama and French 2002; Campbell and Viceira 2002; Gomes and Michaelides 2005). In this subsection I investigate how a lower equity risk premium affect the optimal asset allocation strategy and compare the performances between deterministic lifestyle strategies and the inter-temporal optimal allocations by numerical simulation.

When equity risk premium is 0.04 and RRA=2, the optimal proportions invested in cash and stocks are 0.003151 and 0.996849 respectively for cash-stock scenario; and the optimal proportion invested in bonds and stocks are 0.174696 and 0.825304 respectively for bond-stock scenario. The proportions invested in stocks are lower than those for equity risk premium of 0.06. The results on the expected utility are shown in Table 6. The intertemporal optimal asset allocation with or without short-selling the wage replicating portfolio outperforms the lifestyle strategies. One noticeable result is that the 100% stock strategy outperforms both the lifestyle strategies and the intertemporal optimal asset allocation without short-selling the wage replicating portfolio (“optimal power, non-augmented”) in the two equity-cash scenarios. This result indicates that the 100% stock strategy is closer to the "real" optimal allocation, when a wage replicating portfolio is not used, than the "optimal power, non-augmented" allocation in the two equity-cash scenarios.

Table 6 Expected utility of lifestyle strategies and optimal asset allocation with equity risk premium of 0.04 and RRA=2

Strategy	$\sigma_Y = 0$		$\sigma_Y \neq 0$	
	Equity-cash, γ=2	Equity-bond, γ=2	Equity-cash, , γ=2	Equity-bond, γ=2
t_s=0	-0.17093	-0.14899	-0.17072	-0.13537
t_s=15	-0.15278	-0.13802	-0.15255	-0.13001
t_s=30	-0.13005	-0.12326	-0.12981	-0.12637
Optimal power, augmented wealth	-0.09864	-0.09309	-0.09929	-0.09334
Optimal power, non-augmented	-0.10627	-0.10043	-0.11031	-0.10016
100% cash	-0.22877		-0.22905	
100% bond		-0.19374		-0.19393
100% equity	-0.10621	-0.10621	-0.10629	-0.10629

When equity risk premium is 0.02 and RRA=2, the optimal proportions invested in cash and stocks are 0.277123 and 0.722877 respectively for cash-stock scenario; and the optimal proportion invested in bonds and stocks are 0.3746 and 0.6254 respectively. The proportions invested in stocks are even lower than those for equity risk premium of 0.04. The results on the expected utility are shown in Table 7. The inter-temporal optimal asset allocation with short-selling the wage replicating portfolio ("optimal power, augmented wealth") produces the best outcome, and the inter-temporal optimal asset allocation without short-selling the wage replicating portfolio ("optimal power, non-augmented") produces the second best outcome. The 100% stock strategy does not outperform the inter-temporal optimal asset allocation without short-selling the wage replicating portfolio ("optimal power, non-augmented") in the equity-cash scenarios. The deterministic lifestyle strategy with switching time t_s=0 outperforms the 100% stock strategy in the two equity-bond scenarios. The results in this subsection suggest that a lower equity risk premium reduces the optimal proportion invested in stocks, but the optimal allocation with short-sale of the wage replicating portfolio still produces the best outcome; however, the optimal allocation derived with short-selling the wage replicating portfolio may not be the optimal allocation for the scenario where short-selling wage replicating portfolio is not allowed.

Table 7 Expected utility of lifestyle strategies and optimal asset allocation with equity risk premium of 0.02 and RRA=2

Strategy	$\sigma_Y = 0$		$\sigma_Y \neq 0$	
	Equity-cash, γ=2	Equity-bond, γ=2	Equity-cash, , γ=2	Equity-bond, γ=2
t_s=0	-0.20158	-0.17672	-0.20137	-0.16115
t_s=15	-0.19566	-0.1774	-0.19541	-0.16714
t_s=30	-0.18757	-0.17798	-0.18727	-0.18038
Optimal power, augmented wealth	-0.14957	-0.12458	-0.15002	-0.12502
Optimal power, non-augmented	-0.16982	-0.14198	-0.16997	-0.15961
100% cash	-0.22877		-0.22905	
100% bond		-0.19374		-0.19393
100% equity	-0.17702	-0.17702	-0.17714	-0.17714

4.5. The effects of non-hedgeable wage risk and pension contribution rate on the optimal asset allocation

Although Table 3 and Fig.3 appear to indicate that the optimal proportions solved for the $\sigma_Y = 0$ scenario is still optimal for the $\pi \neq 0$ and $\sigma_Y \neq 0$ scenario, the results in Table 6 suggest the optimal allocation derived with short-selling the wage replicating portfolio may not be the optimal allocation for the scenario where short-selling wage replicating portfolio is not allowed. In this subsection I will explore 1) whether the value of non-hedgeable wage risk (σ_Y) will affect the optimal asset allocation, and 2) whether different pension contribution rate will affect the optimal allocation when short-selling wage replicating portfolio is not used. Since there is no analytical solution for power utility when contribution from wage incomes is not hedged or stopped, these two problems have to be solved by numerical simulations.

Table 8 Optimal proportion in stock for augmented pension wealth

(with parameters in Table 1 except σ_Y and π)

	Equity-cash			Equity-bond		
	$\pi = 0.02$	$\pi = 0.1$	$\pi = 0.5$	$\pi = 0.02$	$\pi = 0.1$	$\pi = 0.5$
$\sigma_Y = 0$	1.29	1.29	1.29	1.04	1.04	1.04
$\sigma_Y = 0.01$	1.29	1.29	1.29	1.04	1.04	1.04
$\sigma_Y = 0.02$	1.29	1.29	1.29	1.04	1.04	1.04
$\sigma_Y = 0.1$	1.29	1.29	1.29	1.04	1.04	1.04
$\sigma_Y = 0.2$	1.29	1.29	1.29	1.04	1.04	1.04

Table 8 summarizes the results for different values of non-hedgeable wage risk (σ_Y) and pension contribution rate (π) when a wage replicating portfolio is short-sold for both $\sigma_Y = 0$ and $\sigma_Y \neq 0$ scenarios (even though the wage income cannot be fully hedgeable for $\sigma_Y \neq 0$). If the investors short-sell the same replicating portfolio of value $-\pi Y(t) f(t)$ according to equation (12) when $\sigma_Y \neq 0$, the optimal proportion invested in stocks for the pension portfolio (excluding the short-sold replicating portfolio) is the same as that when $\sigma_Y = 0$ for pension plans investing in cash and stocks (Table 8). This is also true for pension plans investing in bonds and stocks. The value of pension contribution rates has no effect on the optimal proportion invested in stocks. These results show that the optimal proportion in stocks is independent of the value of σ_Y or the pension contribution rate π (Table 8). It has been shown in the subsection 4.3 that the expected terminal utility is lower in the $\sigma_Y \neq 0$ case than that in the $\sigma_Y = 0$ case for the pension portfolio. Since the replicating portfolio of value

$-\pi Y(t)f(t)$ according to equation (12) may not be exactly paid by the future pension contributions in the $\sigma_Y \neq 0$ case, the expected terminal utility from overall wealth (the sum of pension portfolio and the short-sold replicating portfolio) is lower in the $\sigma_Y \neq 0$ case than that in the $\sigma_Y = 0$ case. The value of optimal proportion in stock by numerical methods is slightly different from the earlier analytical solution, because the number grids are discrete and there are rounding errors in computation.

Table 9 summarizes the results for different values of non-hedgeable wage risk (σ_Y) and pension contribution rate (π) when a wage replicating portfolio is not short-sold for either $\sigma_Y = 0$ or $\sigma_Y \neq 0$ scenarios. If the investors do not augment their initial pension wealth by short-selling the replicating portfolio of value $-\pi Y(t)f(t)$ according to equation (12), the optimal proportion invested in stocks is the same for both $\sigma_Y = 0$ and $\sigma_Y \neq 0$ cases. This is true for pension plans investing in cash and stocks as well as pension plans investing in bonds and stocks (Table 9). The value of σ_Y or the pension contribution rate π does not affect the optimal proportion in stocks. The optimal proportion in stocks without short-selling the replicating portfolio is higher than that with augmented pension wealth (by comparing Table 9 with Table 8), which is consistent with previous studies (Bodie et al 1992; Viceira 2001).

Table 9 "Optimal" proportion in stock for non-augmented pension wealth and wage contributions

(with parameters in Table 1 except σ_Y and π)

	Equity-cash			Equity-bond		
	$\pi = 0.02$	$\pi = 0.1$	$\pi = 0.5$	$\pi = 0.02$	$\pi = 0.1$	$\pi = 0.5$
$\sigma_Y = 0$	1.41	1.41	1.41	1.08	1.08	1.08
$\sigma_Y = 0.01$	1.41	1.41	1.41	1.08	1.08	1.08
$\sigma_Y = 0.02$	1.41	1.41	1.41	1.08	1.08	1.08
$\sigma_Y = 0.1$	1.41	1.41	1.41	1.08	1.08	1.08
$\sigma_Y = 0.2$	1.41	1.41	1.41	1.08	1.08	1.08

The results in this subsection can explain why the 100% stock strategy outperforms the "optimal power, non-augmented" strategy in the equity-cash scenario with an equity risk premium of 0.04 in the preceding subsection 4.4. With a future contribution stream, the optimal proportion in the risky assets is higher than that without a future contribution stream. With an equity risk premium of 0.04, the optimal proportion in the risky asset happens to be more than 100% for the equity-cash scenario with a future contribution stream, whereas the optimal proportion in the risky asset for the equity-cash scenario without a future contribution stream is smaller than 100%. Therefore, the 100% stock strategy is closer to the optimal proportion. From the numerical results in this subsection, we have the following proposition.

Proposition 4: *If the optimal asset allocation strategy for the non-hedgeable wage ($\sigma_Y \neq 0$) case is horizon-independent, it is the same as the optimal asset allocation strategy for the fully hedgeable wage ($\sigma_Y = 0$) case.*

To be accurate, the "optimal" proportions in Table 5 are in fact the best or optimal static allocation. The numerical procedure used here is to search the best static allocation that leads to the highest utility when applied to all time points. Only when the horizon-independence of the optimal strategy for the non-augmented pension wealth and wage contributions has been proved, can these proportions be the true optimal allocation. This paper has not proved the horizon-independence. The numerical procedure for the true optimal allocation will look for the optimal proportion at each time point, which is computationally more demanding. From the studies of Bodie et al (1992) and Viceira (2001), the optimal strategy for the non-augmented pension wealth and wage contributions is likely to be horizon dependent. However, the horizon-dependence might be true for both $\sigma_Y = 0$ and $\sigma_Y \neq 0$ cases if a wage replicating portfolio is not short-sold, and my conjecture is that the optimal allocation is still the same for both $\sigma_Y = 0$ and $\sigma_Y \neq 0$ cases.

5. Conclusion

In this paper I derive optimal asset allocation strategies for power terminal utilities using two assets, cash and stock, or bond and stock, when wage incomes are fully hedgeable. The optimal allocation from the two asset models is used as a benchmark for comparison with deterministic lifestyle asset allocation strategies and with non-hedgeable wage risk scenario.

The deterministic lifestyle strategies can be mimicked by static allocation strategies with same expected return and less risk, and therefore they are second order dominated by their corresponding static allocation strategies. They cannot be the optimal asset allocation strategies for pension funds. A risk-averse investor would prefer a static portfolio with the same expected return and less risk. There is no solution of optimal parameters for deterministic lifestyle strategies that can lead to the same result of inter-temporal optimization in terms of the expected terminal utility. The conditions for optimization cannot be met by choosing a suitable switching time or scheme in deterministic lifestyle strategies.

When terminal utility is a function of wealth-to-wage ratio, a pure cash or pure bond strategy is the most risky, with both a lower expected terminal utility and a higher variance. This result arises because wage growth is more in line with the growth of economy and stock market than risk free interest rate. Therefore, risk free assets are more risky in terms of wealth-to-wage ratio than high risk equities with commonly used market parameters. The deterministic lifestyle strategies produce worse results than inter-temporal optimization or a static high risk 100% equity strategy in numerical simulations. With the commonly assumed market parameters, if there is a constraint on short-sale, the optimal asset allocation strategy would be the 100% equity strategy.

The optimal portfolio composition is independent of the value of non-hedgeable wage risk (σ_Y) and pension contribution rate (π) in the range examined in the present paper. The existence of a future contribution stream increases the optimal proportion invested in the risky asset by the same amount, no matter whether $\sigma_Y = 0$ and $\sigma_Y \neq 0$. As long as the hedgeable part of wage income is hedged by short-selling a replicating portfolio, the optimal asset allocation is the same no matter whether $\sigma_Y = 0$ and $\sigma_Y \neq 0$. An increase in relative risk aversion or a decrease in equity risk premium reduces the optimal proportion invested in the risky asset (stock). The expected terminal utility is lower for $\sigma_Y \neq 0$ case than that for the $\sigma_Y = 0$ case.

In conclusion, the optimal asset allocation strategy from inter-temporal optimization produces higher expected terminal utility than that of deterministic lifestyle strategies. The deterministic lifestyle strategies produce lower expected terminal utility than the 100% equity strategy with commonly assumed equity risk premium. The 100% cash or 100% bond strategy is the least efficient and most risky in long term. The optimal portfolio composition is the same for the $\sigma_Y = 0$ and $\sigma_Y \neq 0$ cases when the pension wealth is augmented by short-selling a replicating portfolio or when contribution from wage income is added to the pension when it comes in.

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Appendix A Solution for two assests with hedgeable wage income (power utility)

The corresponding Hamilton-Jacobi-Bellman equation for the optimal asset allocation problem using two assets is

$$H(J) = J_t + \mu'_w \frac{\partial J}{\partial w} + (\theta M + u)\tilde{X}\frac{\partial J}{\partial \tilde{X}} + \frac{1}{2}tr\left(\Omega'\Omega\frac{\partial^2 J}{\partial w^2}\right) + (\theta\Gamma' + \Lambda')\Omega\tilde{X}\frac{\partial^2 J}{\partial w \partial \tilde{X}}$$
$$+\frac{1}{2}(\theta\Gamma'\Gamma\theta + 2\theta\Gamma'\Lambda + \Lambda'\Lambda)\tilde{X}^2\frac{\partial^2 J}{\partial \tilde{X}^2}$$

(A-1)

The optimal portfolio composition from the first order condition is

$$\theta^* = -(\Gamma'\Gamma)^{-1}\Gamma'\Lambda - (\Gamma'\Gamma)^{-1}M\frac{J_{\tilde{X}}}{\tilde{X}J_{\tilde{X}\tilde{X}}} - (\Gamma'\Gamma)^{-1}\Gamma'\Omega\frac{J_{w\tilde{X}}}{\tilde{X}J_{\tilde{X}\tilde{X}}} \qquad \text{(A-2)}$$

Assuming that the maximized expected terminal utility of plan members has the functional form

$J(t,x,w) = \frac{1}{1-\gamma}g(t,w)^{\gamma}x^{1-\gamma}$,we have

$$J_t = \frac{\gamma}{1-\gamma}g^{\gamma-1}g_t x^{1-\gamma},$$

$$J_x = g^{\gamma}x^{-\gamma},$$

$$J_{xx} = -\gamma g^{\gamma}x^{-\gamma-1},$$

$$J_w = \frac{\gamma}{1-\gamma}g^{\gamma-1}g_w x^{1-\gamma},$$

$$J_{ww} = -\gamma g^{\gamma-2}{g_w}^2 x^{1-\gamma} + \frac{\gamma}{1-\gamma}g^{\gamma-1}g_{ww}x^{1-\gamma},$$

$$J_{xw} = \gamma g^{\gamma-1}g_w x^{-\gamma}. \qquad \text{(A-3)}$$

In the above equations J_w, J_{xw} and g_w are vectors, and J_{ww} and g_{ww} are matrices. Substituting the derivatives of expected terminal power utility function into the above HJB equation gives

$$\frac{\gamma}{1-\gamma} g^{\gamma-1} g_t x^{1-\gamma} + (\theta M + u) g^{\gamma} x^{1-\gamma} + \mu_w ' \frac{\gamma}{1-\gamma} g^{\gamma-1} g_w x^{1-\gamma}$$
$$+ \frac{1}{2} tr \left\{ \Omega ' \Omega \left[-\gamma g^{\gamma-2} g_w{}^2 x^{1-\gamma} + \frac{\gamma}{1-\gamma} g^{\gamma-1} g_{ww} x^{1-\gamma} \right] \right\} + (\theta \Gamma ' + \Lambda ') \Omega \gamma g^{\gamma-1} g_w x^{\gamma}$$
$$+ \frac{1}{2} [\theta \Gamma ' \Gamma \theta + 2 \theta \Gamma ' \Lambda + \Lambda ' \Lambda](-\gamma) g^{\gamma} x^{1-\gamma} = 0$$

(A-4)

Substituting the optimal proportion composition of pension fund investment θ* and simplifying, the above equation becomes

$$g_t + \left[\mu_w ' + \frac{1-\gamma}{\gamma} M '(\Gamma ' \Gamma)^{-1} \Gamma \Omega \right] g_w + \frac{1}{2} tr(\Omega ' \Omega g_{ww})$$
$$- \left[\frac{\gamma}{2(-\gamma)^2} M '(\Gamma ' \Gamma)^{-1} M + \frac{1-\gamma}{\gamma} M '(\Gamma ' \Gamma)^{-1} \Gamma ' \Lambda - \frac{1-\gamma}{\gamma} u \right] g = 0$$

(A-5)

By the Feynman-Kac formula (Øksendal 2000; Duffie 2001), there exists a probability measure $Q(\gamma)$ such that

$$g(t, w(t)) = E_{Q(\gamma)}[g(T, \tilde{w}(T)) D(t,T) \mid F_t], \qquad \text{(A-6)}$$

where $\tilde{w}(s)$ is governed by the SDE

$$d\tilde{w}(s) = \tilde{\mu}_w(\tilde{w}(s))ds + \Omega(\tilde{w}(s), s)' dZ ,$$

$$\tilde{\mu}_w(\tilde{w}(s)) = \mu_w + \frac{1-\gamma}{\gamma} M '(\Gamma ' \Gamma)^{-1} \Gamma ' \Omega ,$$

$$\tilde{w}(t) = w(t) ,$$

and

$$D(t,T) = \exp\left[\int_t^T \varphi(s) ds \right],$$

where

$$\varphi(s) = -\left[\frac{1-\gamma}{2(-\gamma)^2} M '(\Gamma ' \Gamma)^{-1} M + \frac{1-\gamma}{\gamma} M '(\Gamma ' \Gamma)^{-1} \Gamma ' \Lambda - \frac{1-\gamma}{\gamma} u \right].$$

In equation (A-6), F_t is the filtration, which can be interpreted as the information available to the investor at time t. The s in $\varphi(s)$ stands for time, and the function is written as $\varphi(s)$ to indicate that φ might be a function of time (if one or more of the parameters in M, Γ, Λ and u are time dependent). In this paper, all parameters in M, Γ, Λ and u are assumed to be constant, and therefore $\varphi(s)$ are constant.

Using the results from the Feynman-Kac formula, i.e. (A-6), the optimal portfolio composition (equation (A-2)) is

$$\theta^* = -(\Gamma'\Gamma)^{-1}\Gamma'\Lambda - (\Gamma'\Gamma)^{-1} M \frac{1}{-\gamma} + (\Gamma'\Gamma)^{-1}\Gamma'\Omega\int_t^T \frac{\partial}{\partial w_t} E_t[\varphi(s)]ds. \quad \text{(A-7)}$$

Since all the terms in the function $\varphi(s)$ do not depend on the state variables r and Y, its derivatives with respect to w_t are zero and the above equation becomes

$$\theta^* = -(\Gamma'\Gamma)^{-1}\Gamma'\Lambda + (\Gamma'\Gamma)^{-1} M \frac{1}{\gamma}. \quad \text{(A-8)}$$

In equation (A-8), only the second term, i.e. the speculative component, depends on the relative risk aversion γ.

For pension plans investing in cash and stocks, the first item in the above equation is

$$\theta_1^* = -(\Gamma'\Gamma)^{-1}\Gamma'\Lambda = \frac{v_{rY}v_{rS}\sigma_r^{\ 2} + v_{SY}\sigma_S^{\ 2}}{v_{rS}^{\ 2}\sigma_r^{\ 2} + \sigma_S^{\ 2}}$$

The second item is

$$\theta_2^* = (\Gamma'\Gamma)^{-1} M \frac{1}{\gamma} = \frac{m_S\xi - v_{rS}v_{rY}\sigma_r^{\ 2} - v_{SY}\sigma_S^{\ 2}}{\gamma(v_{rS}^{\ 2}\sigma_r^{\ 2} + \sigma_S^{\ 2})}$$

The optimal proportion of pension wealth invested in stocks is

$$\theta^* = \frac{v_{rY}v_{rS}\sigma_r^{\ 2} + v_{SY}\sigma_S^{\ 2}}{v_{rS}^{\ 2}\sigma_r^{\ 2} + \sigma_S^{\ 2}} + \frac{m_S - v_{rS}v_{rY}\sigma_r^{\ 2} - v_{SY}\sigma_S^{\ 2}}{\gamma(v_{rS}^{\ 2}\sigma_r^{\ 2} + \sigma_S^{\ 2})} \quad \text{(A-9)}$$

For pension plans investing in bonds and stocks, the first item in the equation (A-8) is

$$\theta_1^* = -(\Gamma'\Gamma)^{-1}\Gamma'\Lambda = \frac{(b_K + v_{rY})(b_K + v_{rS})\sigma_r^{\ 2} + v_{SY}\sigma_S^{\ 2}}{(b_K + v_{rS})^2\sigma_r^{\ 2} + \sigma_S^{\ 2}}$$

The second item is

$$\theta_2^* = (\Gamma'\Gamma)^{-1} M \frac{1}{\gamma} = \frac{m_S - b_K\sigma_r\xi - (b_K + v_{rS})v_{rY}\sigma_r^{\ 2} - v_{SY}\sigma_S^{\ 2}}{\gamma[(b_K + v_{rS})^2\sigma_r^{\ 2} + \sigma_S^{\ 2}]}$$

The optimal proportion of pension wealth invested in stocks is

$$\theta^* = \frac{(b_K + v_{rY})(b_K + v_{rS})\sigma_r^{\ 2} + v_{SY}\sigma_S^{\ 2}}{(b_K + v_{rS})^2\sigma_r^{\ 2} + \sigma_S^{\ 2}} + \frac{m_S - b_K\sigma_r\xi - (b_K + v_{rS})v_{rY}\sigma_r^{\ 2} - v_{SY}\sigma_S^{\ 2}}{\gamma[(b_K + v_{rS})^2\sigma_r^{\ 2} + \sigma_S^{\ 2}]}$$

(A-10)

The optimal proportion of pension wealth invested in cash or bonds is $1-\theta^*$.