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Integrating Financial and Demographic Longevity Risk Models: An Australian Model for Financial Applications

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Abstract

Since its introduction, the Lee Carter model has been widely adopted as a means of modelling the distribution of projected mortality rates. Increasingly attention is being placed on alternative models and, importantly in the financial and actuarial literature, on models suited to risk management and pricing. Financial economic approaches based on term structure models provide a framework for embedding longevity models into a pricing and risk management framework. They can include traditional actuarial models for the force of mortality as well as multiple risk factor models. The paper develops a stochastic longevity model suitable for financial pricing and risk management applications based on Australian population mortality rates from 1971-2004 for ages 50-99. The model allows for expected changes arising from age and cohort effects and includes multiple stochastic risk factors. The model captures age and time effects and allows for age dependence in the stochastic factors driving longevity improvements. The model provides a good fit to historical data capturing the stochastic trends in mortality improvement at different ages and across time as well as the multivariate dependence structure across ages.

Keywords: longevity, mortality, pricing, risk management
JEL classification: G22, G23, C32, J11

1 Introduction

In life insurance and pensions, mortality models have traditionally been based on life tables using deterministic projections to allow for mortality improvement. Life tables are based on an assumption that future death rates are known with certainty. Dramatic improvements in longevity over the 20th century have shown this approach to be inadequate for the management of mortality and longevity risk. Empirical studies, including for example Lee and Carter (1992) [19], clearly show that longevity risk requires a stochastic modelling approach. Mortality improvement trends have been poorly forecast by deterministic trends. Tuljapurkar (1998) [29] and Tuljapurkar and Boe(1998) [30] review isssues in longevity forecasting.

Life insurers, pension funds and annuity providers are increasingly aware of their exposure to the risk of mortality changes and the need for better models for risk management. They must manage uncertainty in trends and volatility in risk exposures that were previously not formally considered. Actuarial and demographic models for mortality need to be integrated into financial models in order to consider applications such as risk-based capital management and pricing mortality-linked securities. Financial economic approaches based on term structure models, such as that of Dahl (2004) [14], provide a framework for embedding mortality models into a pricing and risk management framework.

Stochastic mortality models were considered by population biologists. Woodbury and Manton (1977) [34] and Yashin et al. (1985) [35] consider mortality as a dynamic process driven by a multivariate state space. An individual's physiological status is determined by multiple factors affecting the health of the individual. Death occurs if the individual enters a state beyond a boundary in the space, corresponding to a "terminal" effect of the risk factors. These models aim to mirror the biological process leading to death. Although promising as a basis for modelling longevity risk, they can involve unobservable risk factors and require large amounts of individual data, some of which is not currently readily available, to calibrate and assess the models.

Stochastic models of mortality is a significant topic of current research in actuarial science and demography. Cairns et al (2007) [9], Macdonald et al, 2003 [22], Currie, Durban and Eilers, 2004 [12], and JP Morgan, 2007 [18], amongst others, develop and assess a range of stochastic models. In the demographic literature stochastic modeling of mortality has been influenced by the modelling approach of Lee and Carter (1992) [19]. They proposed a model for the central rate of mortality $m(x,t)$ as a function of age and time of the form

$$\ln[m(x,t)] = a_x + b_x k_t + \varepsilon_{x,t}. \tag{1}$$

The underlying mortality rate is determined by age specific constants a_x with the evolution of mortality over time driven by a stochastic process k_t which impacts each age differently according to age specific parameters b_x. Short term fluctuations in mortality are modeled

by $\varepsilon_{x,t}$ usually assumed to be normally distributed. The k_t term is modeled using an ARIMA process. The age specific constants a_x and b_x are estimated using a two stage procedure with singular value decomposition. A number of extensions have been proposed to improve the Lee-Carter model. Brouhns, Denuit and Vermunt (2002) [5] fit a Lee-Carter model assuming that the number of deaths follow a Poisson process. Renshaw and Haberman (2003) [24] propose adding additional time varying factors, improving the model fit to a range of ages. Renshaw and Haberman (2006) [25] showed that a cohort effect was required in order to fit gender-based 1961-2003 UK data. Hyndman and Ullah (2007) [17] develop a multi-factor modelling approach using functional principal components to fit demographic data.

The Lee-Carter model has been fitted to mortality data for a range of countries. These studies indicate the need to modify the model to include additional factors or effects such as the cohort effect. The model requires estimation of a large number of parameters and the analysis of Hyndman and Ullah (2007) [17] and Renshaw and Haberman (2003) [24] demonstrate the need for additional stochastic factors. Booth et al (2002) [3] have considered models for Australian data. They fitted the Lee Carter model to Australian data. They find a departure from a pattern of constant exponential decline in the k_t term and non-constant patterns in the age parameter b_x.

The model is also limited in its application to the pricing of mortality-linked securities since the incorporation of a risk adjusted probability measure is not natural. Denuit, Devolder and Goderniaux (2007) [13] use the Wang (2002) [32] transform to risk-adjust mortality rates based on the Lee-Carter model, similar to Lin and Cox (2005) [21]. Cairns et al (2006a) [7] and Bauer and Russ (2006) [1] discuss limitations of the Wang transform when developing a stochastic mortality for pricing and risk management.

Recent approaches to modelling mortality use the framework originally developed to price interest-rate derivatives in continuous time as in the early models of Vasicek (1977) [31] and Cox et al.(1985) [10]. The model framework can ensure mortality rate processes are positive. Mortality linked securities can also be readily priced using this approach. Milevsky and Promislow (2001) [23] developed models for interest rates and mortality in a framework designed to price an annuity option. Dahl (2004) [14] proposed a generalized form for these models, including the Milevsky and Promislow model as a special case. The stochastic diffusion process for the force of mortality at time t for a life initially aged x, denoted by $\mu(x,t)$, is

$$d\mu(t,x) = \alpha^\mu\left(t, x, \mu(t,x)\right) dt + \sigma^\mu\left(t, x, \mu(t,x)\right) dB_t. \tag{2}$$

A special case is the affine models. Both the Ornstein-Uhlenbeck process used by Vasicek (1977) [31] and the Feller process used by Cox et al (1985) [10] are affine. Biffis (2005) [2] introduces jump-diffusion in affine processes. Dahl and Møller (2005) [15] assume a time-inhomogeneous Cox, Ingersoll Ross mortality model. Schrager (2006) [26] develops a multiple factor model for mortality using an affine model for all ages simultaneously and

fits the model to Dutch data.

This paper develops a stochastic longevity model based on Australian population mortality data from 1971-2004 and for ages 50-99 that is designed for financial pricing and risk management applications. The model allows for expected changes arising from age and cohort effects and for multiple stochastic risk factors impacting on longevity. Population mortality is used because it is independently constructed, readily available, and suitable for longevity risk modeling and the development of financial risk management products. Ages above 55 are used since these are the ages where longevity risk has the most financial impact. Data was obtained from the Human Mortality Database.

The model structure has a similar form to that used for term structure models allowing it to be calibrated as a market consistent valuation model for longevity risk. A price of mortality risk is formally included in the model. The model provides a good fit to historical data for the ages and time period considered, capturing the stochastic trends in mortality improvement at different ages, across time as well as the multivariate dependence structure across ages. Wills and Sherris (2008) [33] calibrate the model to insurance linked market prices and analyse the pricing and structuring of longevity bonds.

The remainder of this paper is structured as follows. Section 2 outlines the multivariate mortality model including the risk adjustment for pricing mortality-linked securities. Section 3 summarises the approach used to estimate parameters and provides an assessment of the model based on observed Australian population mortality data. Results from simulations using the model are analysed in Section 4. Section 5 concludes.

2   Multivariate Mortality Model

The model developed in this paper is in the same spirit as the financial modelling approach of Dahl (2004) [14] and Schrager (2006) [26]. The model integrates financial and demographic longevity risk models in a framework suitable for pricing and risk management of longevity risk. Demographic trends are modelled in the expected changes for mortality and all ages are modelled simultaneously using multiple dependent factors to drive the shocks to mortality rates. The model allows expected changes in mortality to vary by age and time. Multiple random factors capture dependence between ages. Since mortality improvement is observed to be similar for individuals of similar ages and for cohorts of individuals, it is important to capture age dependence in the model.

The model is designed to be in a form that can readily be used to price longevity-linked securities based on cash flows from annuity portfolios for lives of multiple ages. Portfolio cash flows and the payments on securitized products need to be aggregated over a range of ages. This aggregation of age dependent cash flows is influenced by mortality dependence

between ages. Apart from Schrager (2006) [26] models currently proposed consider single ages independently. Since pricing requires a change of measure from the real world probabilities to a pricing measure, the model structure must readily allow incorporation of a price of risk.

Motivated by Lee and Carter (1992) [19], Dahl (2004) [14] and Schrager (2006) [26], the model is based on an assumed force of mortality $\mu(x,t)$ for age x at time t of the form

$$\mu(x,t) = \mu(x,0)\exp\left[(a+f(x)+g(t))t + \sigma(x,t)W(x,t)\right]$$
$$0 < x < \omega, \ 0 < t < \omega - x. \tag{3}$$

where a is a constant, $f(x)$ is a deterministic function of age, $g(t)$ is a deterministic function of time, $\sigma(x,t)$ is a deterministic (volatility) function of age and time and $W(x,t)$, for x continuous, is an (infinite dimensional) Brownian motion. For any given age x_k

$$\ln\left[\frac{\mu(x_k,t)}{\mu(x_k,0)}\right] = (a+f(x_k)+g(t))t + \sigma W(t). \tag{4}$$

so that the expected rate of change of $\mu(x,t)$, for any given age, consists of a constant a, plus a component that varies with age $f(x_k)$, plus a time varying component $g(t)$. The age dependent component $f(x_k)$ is similar to the age parameter in the Lee-Carter model. At time t, the variance of this rate of change is σt, which is independent of age and constant per unit of time.

The infinite dimensional system in Equation (3) is made tractable by considering a finite dimensional multivariate random vector of mortality rates with length N, for ages $x = x_1, \ldots, x_N$:

$$\underline{\mu}(t) = \left[\begin{array}{c} \mu(x_1,t) \\ \vdots \\ \mu(x_N,t) \end{array}\right]. \tag{5}$$

The dynamics $\underline{d\mu}(t) = [d\mu(x_1,t), \ldots, d\mu(x_N,t)]'$ are driven by the multivariate Wiener process $\underline{dW}(t)$ with mean 0 and instantaneous covariance matrix Σ:

$$\underline{dW}(t) = \left[\begin{array}{c} dW(x_1,t) \\ \vdots \\ dW(x_N,t) \end{array}\right]. \tag{6}$$

To give the model a form suitable for simulation and estimation, the multivariate Wiener process $\underline{dW}(x,t)$ is expressed in terms of an N-dimensional random vector of independent Wiener processes: $\underline{dZ}(t) = [dZ_1(t), \ldots, dZ_N(t)]'$. The $dW(x,t)$ can be expressed as a linear combination of $dZ(t)$ using a deterministic and constant matrix D as $\underline{dW}(x,t) = D\underline{dZ}(t)$ where

$$D = \left[\begin{array}{ccc} \delta_{11} & \ldots & \delta_{1N} \\ \vdots & \ddots & \vdots \\ \delta_{N1} & \ldots & \delta_{NN} \end{array}\right], \tag{7}$$

and with each element given by

$$dW(x,t) = \sum_{i=1}^{N} \delta_{xi} dZ_i(t) \text{ for } x = x_1, \ldots, x_N : . \tag{8}$$

Each row of D has unit length:

$$||\delta_x|| = \sqrt{\sum_{i=1}^{N} \delta_{xi}^2} = 1 \text{ for all } x. \tag{9}$$

The (n, m) element of the covariance matrix Σ is then given by:

$$
\begin{aligned}
Cov\Big(dW(x_n,t), dW(x_m,t)\Big) &= \left[\sum_{i=1}^{N} \delta_{ni}\delta_{im}\right] Var\Big(dZ_i(t)\Big) \\
&= \sum_{i=1}^{N} \delta_{ni}\delta_{im} dt.
\end{aligned}
\tag{10}
$$

and Σ can be written as:

$$
\begin{aligned}
\Sigma &= \left(D\sqrt{dt}\right)\left(D\sqrt{dt}\right)', \\
&= R_\Sigma dt.
\end{aligned}
\tag{11}
$$

where R_Σ is the associated correlation matrix of Σ and D is the Cholesky decomposition of Σ.

2.1   Model Parameterization

Based on empirical studies including Cairns et al (2007) [9], Renshaw and Haberman (2006) [25] and JP Morgan, 2007 [18], along with preliminary analysis of the Australian data the model in Equation (3) is parameterized as:

$$d\mu(x,t) = \Big(a(x+t) + b\Big)\mu(x,t)dt + \sigma\mu(x,t)dW(x,t) \text{ for all } x. \tag{12}$$

The model in Equation (12) has a drift parameter as an affine function of the current age $(x+t)$. The percentage volatility is a constant σ, so that the variability of $d\mu(x,t)$ increases with $\mu(x,t)$. As $\mu(x,t)$ is an increasing function of the current age $(x+t)$, the process becomes more variable for higher initial ages x, and later times t. Mean reversion is not included in the mortality changes based on the assumption that longevity changes do not revert to a long run mean. The absence of mean reversion is consistent with the approach taken by Liao, Yang and Huang (2007) [20] in their longevity model.

The proposed model based on Equation (12) becomes a system of equations:

$$d\mu(x,t) = \Big(a(x+t)+b\Big)\mu(x,t)dt + \sigma\mu(x,t)\sum_{i=1}^{N}\delta_{x,i}dZ_i(t) \text{ for } x = x_1, \ldots, x_N < \omega. \quad (13)$$

where the dependence between ages is captured in the $\delta_{x,i}$ terms.

2.2   Risk Adjusted Pricing Measure

As in Dahl (2004) [14], Dahl and Moeller (2005) [15] and Cairns et al (2006a) [7], an important application for the model is for pricing mortality-linked securities. To do this the mortality dynamics are derived under an equivalent (risk-adjusted) probability measure. The mortality process will not be a martingale under this measure. Rather, the price process of each security will be a martingale. The longevity risk market is inherently incomplete and as a result the choice of a risk-adjusted measure for mortality is not unique.

The mortality rate for initial age x, $\mu(x,t)$, follows a stochastic diffusion process defined in Equation (12)

$$d\mu(x,t) = (a(x+t)+b)\,\mu(x,t)dt + \sigma\mu(x,t)dW(x,t),$$
$$dW(x,t) = \sum_{i=1}^{N}\delta_{xi}dZ_i(t).$$

on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathbb{P}$ is the 'real-world' probability measure.

From the Cameron-Martin-Girsanov Theorem, see for example Cairns (2004) [6] for more details, the process $dW(x,t)$ under an equivalent (risk-adjusted) probability measure $\mathbb{Q}$ is given by

$$dW^{\mathbb{Q}}(x,t) = \sum_{i=1}^{N}\delta_{xi}\left(dZ_i(t) + \lambda_i(t)dt\right)$$
$$= dW(x,t) + \sum_{i=1}^{N}\delta_{xi}\lambda_i(t)dt. \quad (14)$$

which can be written as

$$\underline{dW}^{\mathbb{Q}}(t) = \underline{dW}(t) + D\underline{\lambda}(t)dt, \quad (15)$$

where $\underline{\lambda}(t) = [\lambda_1(t), \ldots, \lambda_N(t)]'$.

The mortality dynamics for a specific age x under the risk adjusted measure $\mathbb{Q}$ is given

by:

$$d\mu^{\mathbb{Q}}(x,t) = \left(a(x+t)+b\right)\mu^{\mathbb{Q}}(x,t)dt + \sigma\mu^{\mathbb{Q}}(x,t)dW^{\mathbb{Q}}(x,t)$$

$$= \left(a(x+t)+b\right)\mu^{\mathbb{Q}}(x,t)dt + \sigma\mu^{\mathbb{Q}}(x,t)\left(dW(x,t) + \sum_{i=1}^{N}\delta_{xi}\lambda_i(t)dt\right) \qquad (16)$$

$$= \left(a(x+t)+b+\sum_{i=1}^{N}\delta_{xi}\lambda_i(t)\right)\mu^{\mathbb{Q}}(x,t)dt + \sigma\mu^{\mathbb{Q}}(x,t)dW(x,t).$$

This is equivalent to the original process under $\mathbb{P}$, with an additional drift adjustment given by $\left(\sum_{i=1}^{N}\delta_{xi}\lambda_i(t)\right)$.

The choice of risk adjusted measure $\mathbb{Q}$, and thus $\underline{\lambda}$ is not unique and can be derived from market equilibrium utility functions as in Cox et al (1985) [10]. Price data from insurance linked securities can be used to calibrate the market price of risk.

3 Data and Methodology

The model is estimated using Australian Population Mortality Data for ages 50-99 from 1971-2004. The ages are chosen to reflect the exposure of an annuity portfolio. Data is obtained from the Human Mortality Database, University of California, Berkeley (USA), and Max Planck Institute for Demographic Research (Germany). Define $m(x,t)$ as the observed central death rate in year t for lives initially aged x as

$$m(x,t) = \frac{\#\text{ deaths during calendar year } t \text{ aged } (x+t) \text{ last birthday}}{\text{avg. pop. during calendar year } t \text{ aged } (x+t) \text{ last birthday}}. \qquad (17)$$

The data lists the number of deaths per calendar year by 'lower age limit', or age last birthday. Population data is provided for the number of lives alive at age $(x+t)$ at the 30th of June each year. This serves as an estimate for the required population in Equation (17). The observed central death rate can thus be calculated directly from the data.

Calibrating the model requires estimates of the observed force of mortality $\hat{\mu}(x,t)$. Under the assumptions that the force of mortality is constant over each integer age and calendar year so that $\mu(x+u,t+s) = \mu(x,t)$ for integers x and t and all $0 <= (s,u) <= 1$, and the size of the population at all ages remains constant over the calendar year it follows that

$$\hat{\mu}(x,t) = m(x,t). \qquad (18)$$

The observed male and female mortality rates used in this analysis are displayed in Figures (1) and (2).
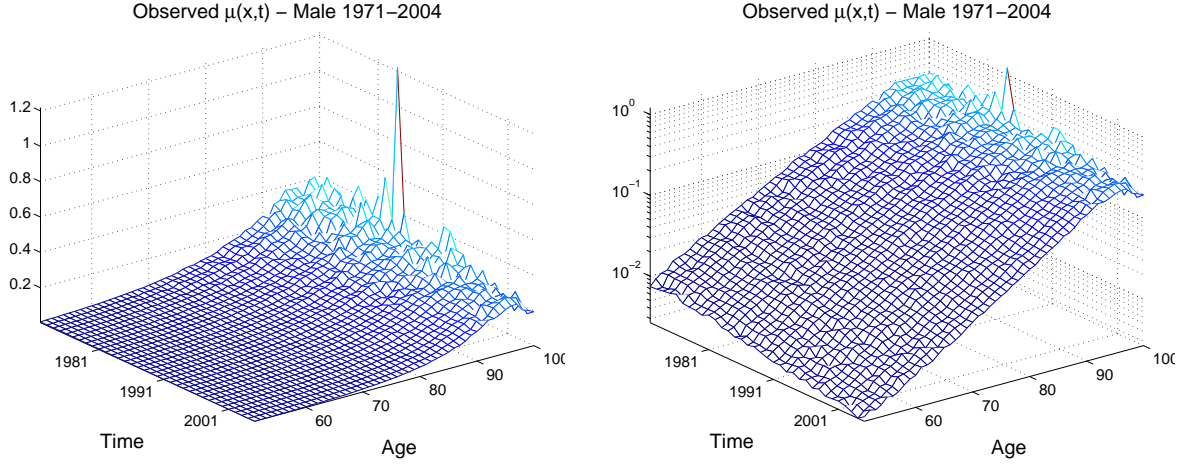
Figure 1: Observed Australian male mortality $\hat{\mu}(x,t)$: 1971-2004, on linear (left) and log (right) scales.

The data shows a linear drift in the logarithm of the mortality rates ages above 50 which is consistent with the model parametrization.

From the estimates $\hat{\mu}(x,t)$, estimates for the change in the force of mortality $\Delta\hat{\mu}(x,t)$ are given by

$$\Delta\hat{\mu}(x,t) = \hat{\mu}(x+1, t+1) - \hat{\mu}(x,t) \tag{19}$$

A matrix of annual observed changes in mortality rates $\Delta\hat{\mu}(x,t)$ is then given by:

$$\Delta\hat{\mu}(x,t) = \begin{bmatrix} \hat{\mu}_{51,1972} & \cdots & \hat{\mu}_{51,2004} \\ \vdots & \ddots & \vdots \\ \hat{\mu}_{99,1972} & \cdots & \hat{\mu}_{99,2004} \end{bmatrix} - \begin{bmatrix} \hat{\mu}_{50,1971} & \cdots & \hat{\mu}_{50,2003} \\ \vdots & \ddots & \vdots \\ \hat{\mu}_{98,1971} & \cdots & \hat{\mu}_{98,2003} \end{bmatrix}. \tag{20}$$

The model parameters are estimated using this data for all elements in the vector $\underline{\Delta\mu}(t)$.

3.1 Parameter Estimation - Maximum Likelihood

Maximum likelihood is used to estimate the parameters of the $d\mu(x,t)$ process:

$$d\mu(x,t) = (a(x+t)+b)\mu(x,t)dt + \sigma\mu(x,t)dW(t).$$

Since

$$\Delta\mu \sim N\left((a(x+t)+b)\mu, \sigma\mu\right).$$

the log-likelihood function is:

$$\ell\left(d\hat{\mu} \mid a,b,\sigma\right) = -\sum_{\text{all } x,t} \ln\left(\sigma\hat{\mu}\sqrt{2\pi}\right) - \frac{1}{2}\sum_{\text{all } x,t} \left(\frac{(\Delta\hat{\mu}/\hat{\mu}) - (a(x+t)+b)}{\sigma}\right)^2,$$
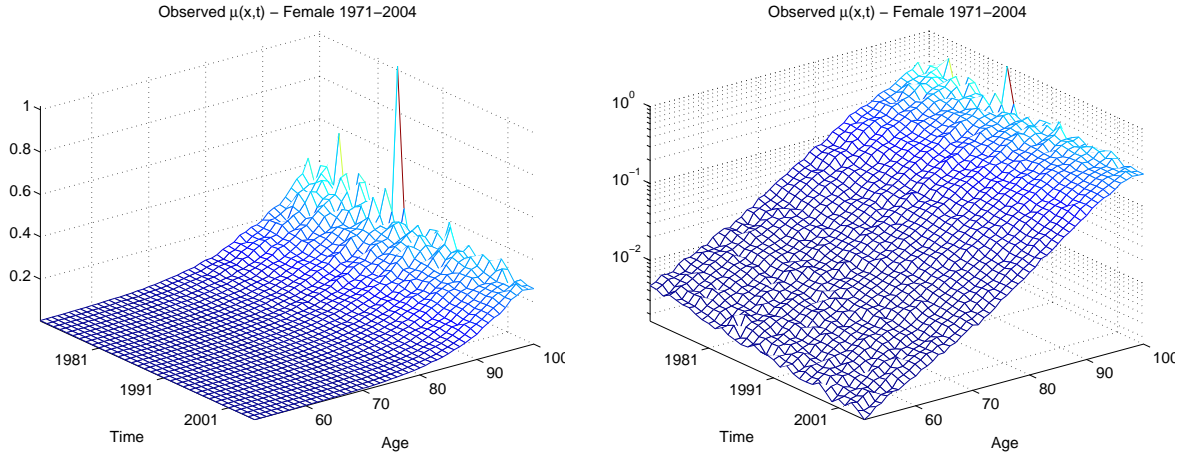
Figure 2: Observed Australian female mortality $\hat{\mu}(x,t)$: 1971-2004, on linear (left) and log (right) scales.

assuming independent and identically distributed sample data, given a, b and σ. Differentiating the log-likelihood function with respect to a gives

$$\frac{\partial \ell}{\partial a} = \sum_{\text{all } x, t} \left(\frac{(\Delta \hat{\mu}/\hat{\mu}) - (a(x+t) + b)}{\sigma} \right) \frac{(x+t)}{\sigma}$$

$$0 = \frac{1}{\sigma^2} \left(\sum_{\text{all } x, t} \frac{\Delta \hat{\mu}(x+t)}{\hat{\mu}} - \hat{a} \sum_{\text{all } x, t} (x+t)^2 - \hat{b} \sum_{\text{all } x, t} (x+t) \right)$$

Differentiating with respect to b gives

$$\frac{\partial \ell}{\partial b} = \sum_{\text{all } x, t} \left(\frac{(\Delta \hat{\mu}/\hat{\mu}) - (a(x+t) + b)}{\sigma} \right) \frac{1}{\sigma}$$

$$0 = \frac{1}{\sigma^2} \left(\sum_{\text{all } x, t} \frac{\Delta \hat{\mu}}{\hat{\mu}} - \hat{a} \sum_{\text{all } x, t} (x+t) - \hat{b}(N \times T) \right)$$

where $(N \times T)$ is the number of observations of $d\hat{\mu}$ across N ages and for T time periods. Differentiating with respect to σ gives

$$\frac{\partial \ell}{\partial \sigma} = -\frac{(N \times T)}{\sigma} + \sigma^{-3} \sum_{\text{all } x, t} ((\Delta \hat{\mu}/\hat{\mu}) - (a(x+t) + b))^2$$

$$\frac{(N \times T)}{\hat{\sigma}} = \hat{\sigma}^{-3} \sum_{\text{all } x, t} \left((\Delta \hat{\mu}/\hat{\mu}) - (\hat{a}(x+t) + \hat{b}) \right)^2$$

$$\hat{\sigma} = \sqrt{\frac{\sum_{\text{all } x, t} \left((\Delta \hat{\mu}/\hat{\mu}) - (\hat{a}(x+t) + \hat{b}) \right)^2}{(N \times T)}}.$$

Parameter	MLE: Male	MLE: Female
$\hat{a}$	-9.4398E-04	2.6993E-04
$\hat{b}$	0.1347	0.0608
$\hat{\sigma}$	0.0906	0.0873

Table 1: Parameter estimates for the mortality model using MLE.

Simultaneously solving the expressions for $\hat{a}$, $\hat{b}$, $\hat{\sigma}$ yields their Maximum Likelihood Estimates. The parameter estimates are summarized in Table 1.

The asymptotic variance/covariance matrix of the maximum likelihood estimates is given in Table 2. For both male and female data, these values are small, indicating a high level of confidence in the estimates. The large sample distribution of the maximum likelihood estimates $\underline{p}$ is asymptotically normally distributed, with mean $\bar{\underline{p}}$ and variance:

$$Var(\underline{p}) = \frac{1}{nI(\bar{p})}, \tag{21}$$

where $\bar{p}$ is the vector of true parameters. This variance is determined using the approximation $I(\bar{\underline{p}}) = I(\underline{p})$. The number of observations is $n = 1617$. $I(\underline{p})$ is found by shocking each parameter and noting the change in $\frac{\partial}{\partial p_i} \log L(X|\underline{p})$ for all p_i. Note that $\underline{p} = [p_1, \ldots, p_n]'$ is the vector of maximum likelihood estimates for data set X and $I(p)$ is the matrix with ij^{th} component:

$$I(\underline{p})_{ij} = -E\left[\frac{\partial^2}{\partial p_i \partial p_j} \log L(X|\underline{p})\right], \tag{22}$$

where $L(X|\underline{p})$ is the likelihood function of X given parameters $\underline{p}$.

	Male				Female		
	a	b	σ		a	b	σ
a	5.53E-13	4.24E-11	-	a	5.13E-13	3.94E-11	-
b	4.24E-11	3.14E-09	-	b	3.94E-11	2.91E-09	-
σ	5.01E-11	2.84E-07	1.61E-09	σ	-4.48E-11	-2.54E-07	1.50E-09

Table 2: Asymptotic var/covar. matrix for male and female MLEs.

3.2 Dependence and Principal Components Analysis

Dependence in the mortality process $\underline{d\mu}(t)$ is modelled based on dependence in the multivariate Wiener process:

$$\underline{dW}(t) = \begin{bmatrix} dW(x_1, t) \\ \vdots \\ dW(x_N, t) \end{bmatrix}. \tag{23}$$

The model assumes that $\underline{dW}(t)$ are independent between time periods. Across a single time period, $\underline{dW}(t)$ incorporates dependence between each age $x = x_1, \ldots, x_N$. To estimate the dependence the residuals are taken from the model with $\hat{a}$, $\hat{b}$, $\hat{\sigma}$ based on their Maximum Likelihood Estimates to get estimated standardised residuals:

$$r(x, t) = \frac{\Delta\hat{\mu}(x, t)/\mu(\hat{x}, t) - (\hat{a}(x + t) + \hat{b})}{\hat{\sigma}}. \tag{24}$$

The standardized residual values for each year are a realization of the random vector $\underline{dW}(t)$. These $r(x, t)$ are used to estimate the sample covariance matrix $\hat{\Sigma}$ of $\underline{dW}(t)$. The standardised residuals are jointly normally distributed $\underline{dW}(t) \sim M.V.N(0, \Sigma)$ where Σ is the associated $N \times N$ positive semi-definite covariance matrix.

Principal components analysis (PCA) is applied to the standardised residuals. Principal Components Analysis (PCA) provides a method for generating random samples of a correlated random vector. These samples are designed to have the same covariance properties as an observed data set. Define $\underline{\theta} = [\theta_1, \ldots, \theta_N]'$ as the ordered eigenvalues of Σ, with $\theta_1 >= \ldots >= \theta_N$. The corresponding eigenvectors are given by the matrix $V = [V_1, \ldots, V_N]$ such that

$$\Sigma V = VT$$
$$\Sigma = VTV',$$

where T is the $N \times N$ diagonal matrix with diagonal $\underline{\theta}$. The term $(V\sqrt{T})$ is the Cholesky decomposition of Σ such that:

$$V\sqrt{T} = D\sqrt{dt} \tag{25}$$

By generating independent normal random samples of $\underline{\eta}$ we are able to use the linear transformation in Equation (28) to generate dependent random samples $\underline{\nu}$ of the variable $\underline{dW}(t)$ at each time period.

The observed age-based correlation matrix $\hat{\Sigma}$ has a total of 49 eigenvectors. Table 3 summarises the percentage of the observed variation explained by these vectors. Hyndman and Ullah (2007) [17] and Hyndman and Booth (2006) [16] consider $(\mu(x, t) - \hat{\mu}(x))$, where $\mu(x, t)$ is the observed mortality rate and $\hat{\mu}(x)$ is the average mortality rate for age x across all years. Principal components analysis is employed to determine the b_x and k_t terms in a

Lee-Carter model. They find that six factors are sufficient to explain the variability in the mortality process, with 90% accounted for by the first factor. They do not include a time trend and do not examine the changes in the standardised residuals. This is expected to result in the dominance of the first eigenvector. The analysis presented here is based on the standardised residuals after allowing for a time trend and using the MLE parameter estimates $\hat{a}$, $\hat{b}$ and $\hat{\sigma}$ to standardise the residuals.

# of Eigenvectors	% of Observed Variation
1	29.3%
5	69.8%
10	85.1%
15	92.4%
20	96.5%
25	97.1%
30	99.1%
31	99.5%
32	100.0%

Table 3: Percentage of the observed variation in residuals explained by the eigenvectors using PCA.

3.3 Analysis of Fit

The model for mortality is found to fit observed Australian population data for the age range $x = 50, \ldots, 99$ and time period $t = 1971, \ldots, 2004$ very well. The analysis of fit is based on the standardised residuals from the model, defined in Equation (24). Our analysis relies on the assumption that the residuals are normally distributed with mean 0 and variance 1. These residuals are illustrated in Figure 3.

There are no trends in either the age or the time dimension, and they are randomly distributed around zero. The model fit is confirmed by the residual descriptive statistics summarized in Table 4. The statistics further show that standard error of the mean estimate is small, and the standard deviation of the residuals is very close to 1.

Pearson's chi-square statistic for quantifying the discrepancy between observed (O) and expected (E)

$$X^2 = \sum_{\text{all obs.}} \frac{(O_i - E_i)^2}{E_i} \tag{26}$$

has approximately a chi-square distribution with degrees of freedom:

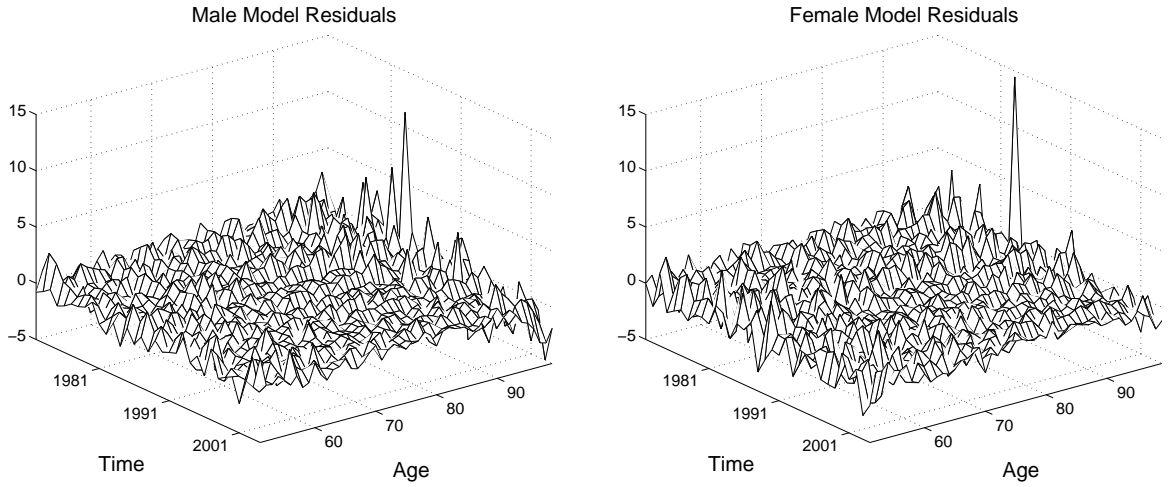$$df = \text{number of observations} - \text{number of independent parameters} - 1 \tag{27}$$

Figure 3: Male and female fitted residuals.

	Male	Female
Mean	-3.12E-03	4.84E-08
Standard Error	0.024	0.025
Standard Deviation	0.972	1.000
Minimum	-6.320	-4.396
Maximum	6.277	14.459
Confidence Level(95.0%)	0.047	0.049

Table 4: Residual descriptive statistics for male and female data.

There are 50 ages $\times$ 34 years of observed $\mu(x,t)$ values. The corresponding number of observations for $d\mu(x,t)$ is $49 \times 33 = 1617$. There are 3 estimated parameters by MLE, and 1225 independent parameters for the 49×49 correlation matrix of $dW(t)$. This metric generates the values given in Table 5. Higher values of the statistic X^2 suggest a poorer fit, and larger deviance of observed values from those expected. This is compared to the chi-square distribution with 388 degrees of freedom. This confirms the model provides a very good fit to the data.

X^2 Male	X^2 Female	χ^2_{388} at 99%
71.08	23.16	326.15

Table 5: Pearson's chi-square statistic and 99% confidence level.

Figure 4 illustrates Male and Female residual histograms from an illustrative subset of the data. The distribution of the residuals at each time is approximately normal with zero mean. This provides the basis for the simulations discussed below.
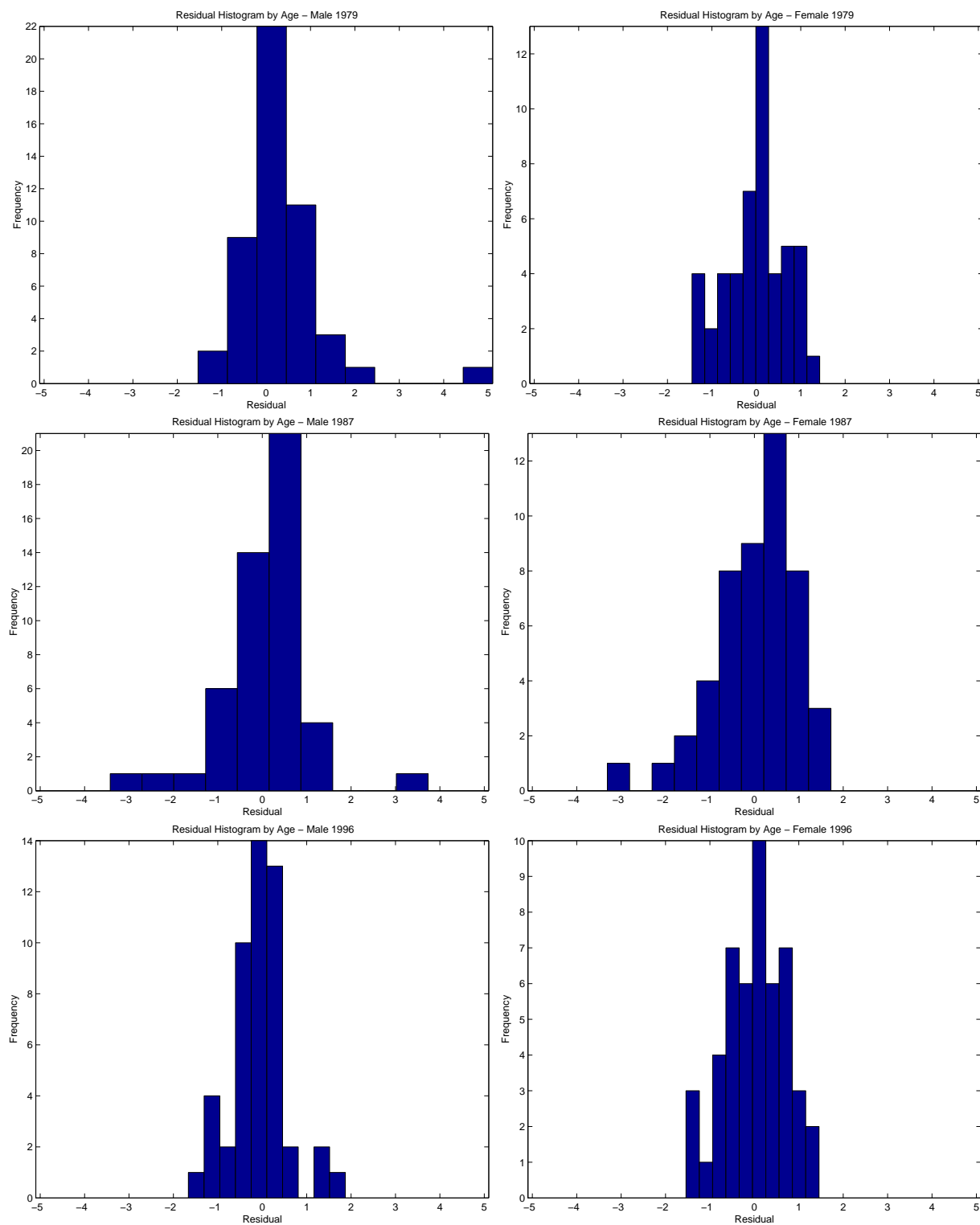
Figure 4: A selection of residual histograms for male and female data, at specific times (1979, 1987 and 1996).

4 Model Simulation

To illustrate the application of the model and its distributional properties the model is simulated. For the purpose of illustration 31 of the 49 eigenvectors have been used although a lower number of around 10 would also be reasonable. The simulation is for a random vector $\underline{\nu}$ and its linear transformation:

$$\underline{\nu} = V\sqrt{T}\underline{\eta}, \tag{28}$$

where $\sqrt{T}$ is an $N \times N$ diagonal matrix with diagonal $\sqrt{\underline{\theta}} = [\sqrt{\theta_1}, \ldots, \sqrt{\theta_N}]'$. The N-dimensional random vector $\underline{\eta} \sim iid\ N(0, I)$, where I is an $N \times N$ unit diagonal matrix. The vector $\underline{\nu}$ will in turn be normally distributed with covariance matrix:

$$\Sigma_\nu = (V\sqrt{T})I(V\sqrt{T})' \tag{29}$$
$$= VTV' \tag{30}$$
$$= \Sigma. \tag{31}$$

Σ is the covariance matrix of our original residual vector $\underline{dW}(t)$.

For each simulation of $\mu(x, t)$, the number alive at each time for a given cohort, x, is given by:

$$l(x, t) = l(x, 0)\exp[\int_0^t \mu(x, s)ds]. \tag{32}$$

Twenty year projections of expected mortality are illustrated in Figure 5. Mortality is projected by tracing the change in mortality rates for each initial cohort through time. The effect of this is seen in the diagonal trends depicted in Figure 5. These trace the progress of the cohort through the age/time space. The 'ridges' in the projection are a result of a cohort's higher observed mortality in the base year (2004).

Figure 6 illustrates the observed and fitted residuals under three different assumptions on the correlations between ages. The first two plots illustrate the observed and fitted residuals for Australian male data using 100,000 simulations. Each line illustrates the residual values at a different time/simulation. The second two plots are based on female data. The final plots are simulations generated under perfect independence and dependence (left to right) between ages at each time.

The distribution of future mortality rates can be seen in Figure 7. This provides 95% confidence intervals for the projected mortality curve, at two points in time, under three types of age-dependence. As a means of reference, the base (2004) mortality curve is also included. The impact of age-dependence is illustrated by the spread of the confidence bounds. Over the majority of ages, mortality projections are more constrained under observed dependence than under either perfect dependence or independence. However,
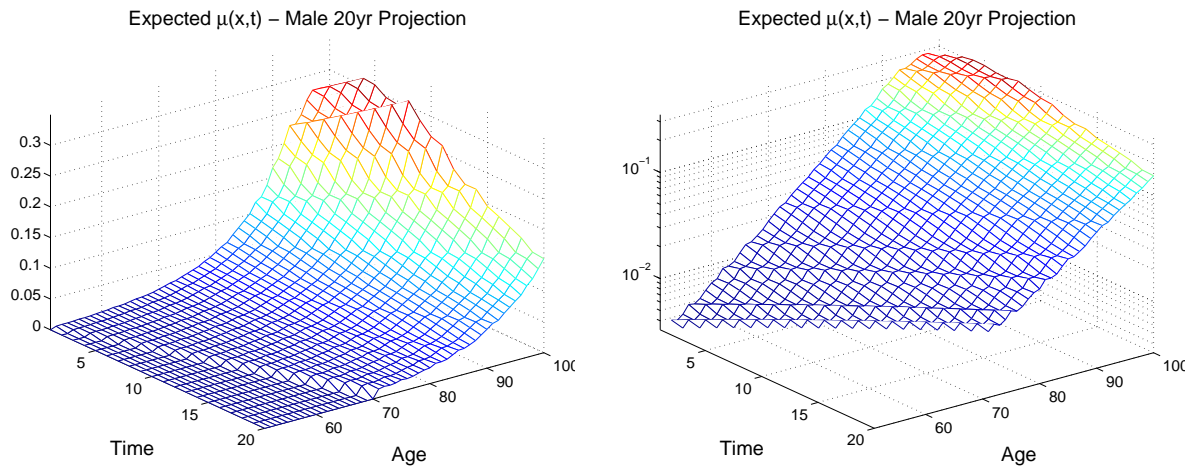
Figure 5: A 20 year projection of male expected mortality (linear and log scales).

above age 95 the observed structure is more volatile. These results are consistent with the work of Hyndman and Booth (2006) [16], who project Australian mortality using the Lee-Carter method.

Figure 8 shows the expected number alive $E[l(x,t)]$ for each age and time, given $l(x,0) = 2,000 \forall x$. The increase at age 99 for periods in the near future is a result of the downturn in the observed force of mortality during the base year. From the plot it can be seen that mortality is expected to improve at all ages.

Collectively, these results show that the proposed model provides a very good fit to the observed data. The assumption that the volatility in the population's size is primarily driven by the underlying mortality process is found to be consistent with experience. Mortality is projected to continue to improve at a rate consistent with past trends, and the projected Australian mortality rates are consistent with those given by Hyndman and Booth (2006), using the Lee Carter model.

Mortality rates at ages 95-99 are projected to increase in the near future, before assuming a general downward trend. This is attributed to the fitting of the model to the most recent 30 years of data. Over this period, old age mortality has experienced significant fluctuations (see observed mortality data in Figure 1), and in recent years has decreased at the highest ages. This has been inconsistent with prior years and female data. Reasons for the decrease may be age specific volatility, or a fundamental shift in the mortality curve.
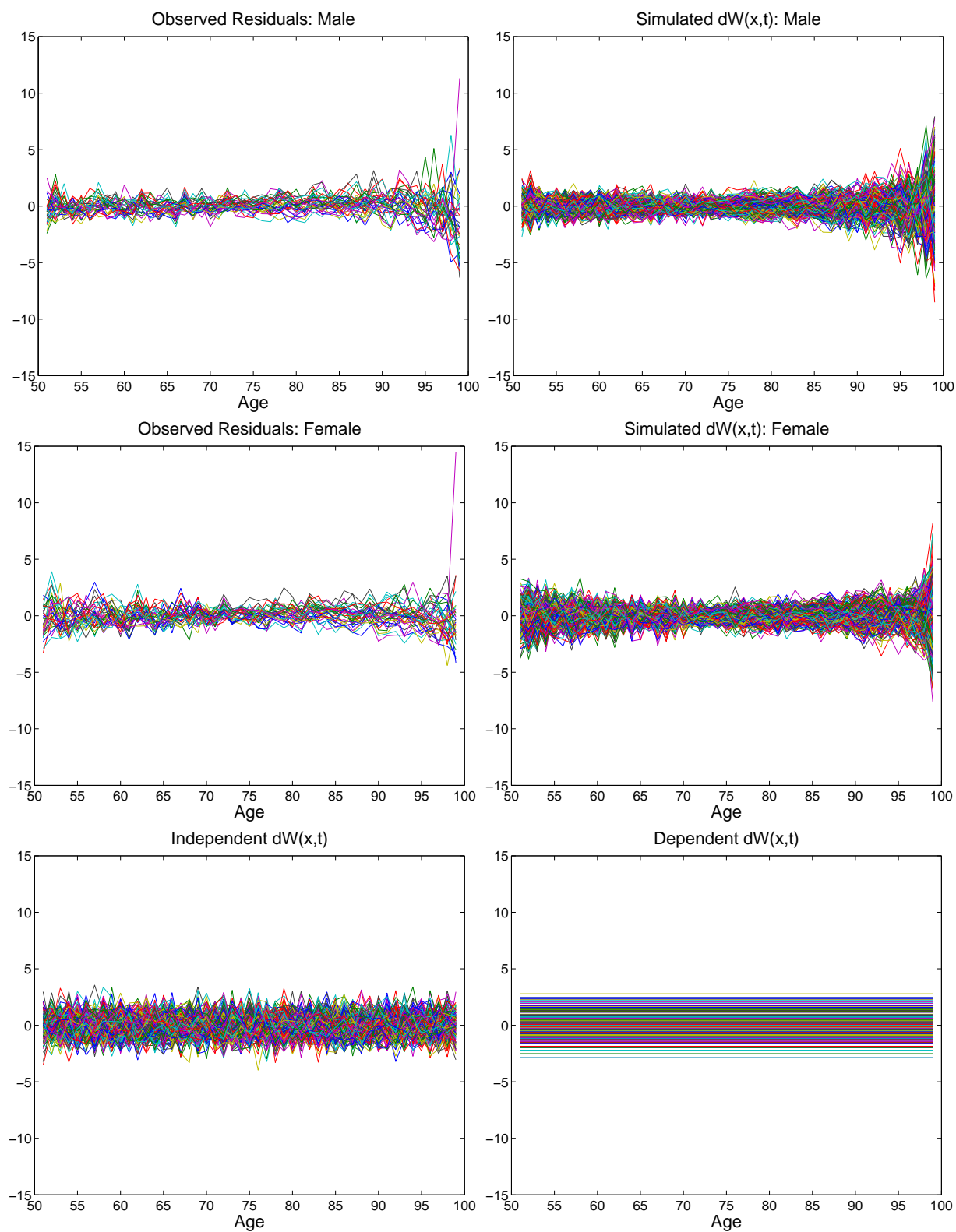
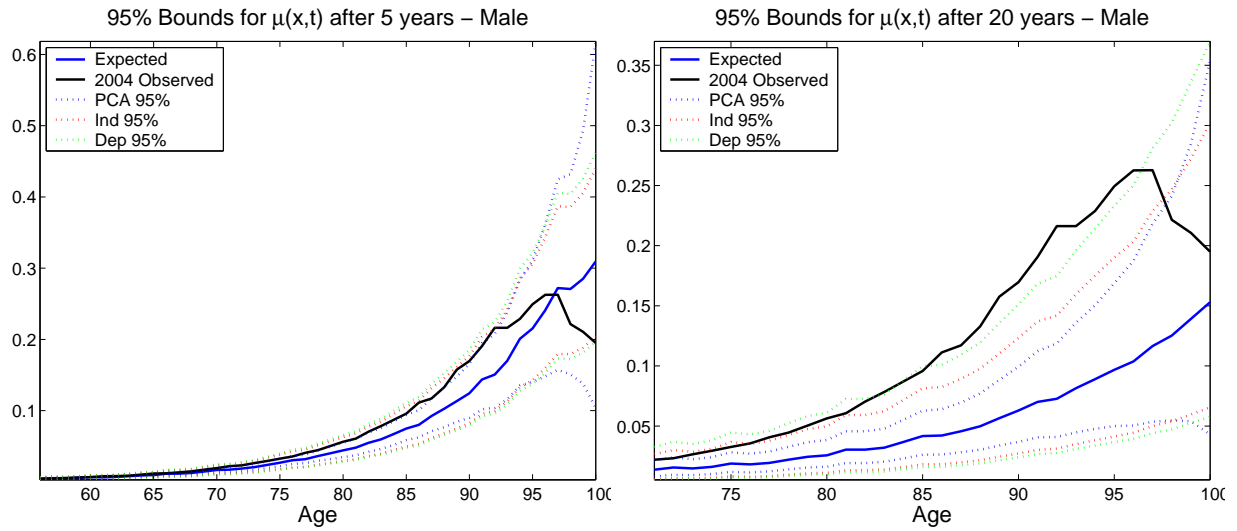**Figure 6:** Observed and fitted residuals under three age-dependence assumptions.

Figure 7: 95% confidence intervals for projected male mortality, at 5 years and 20 years.
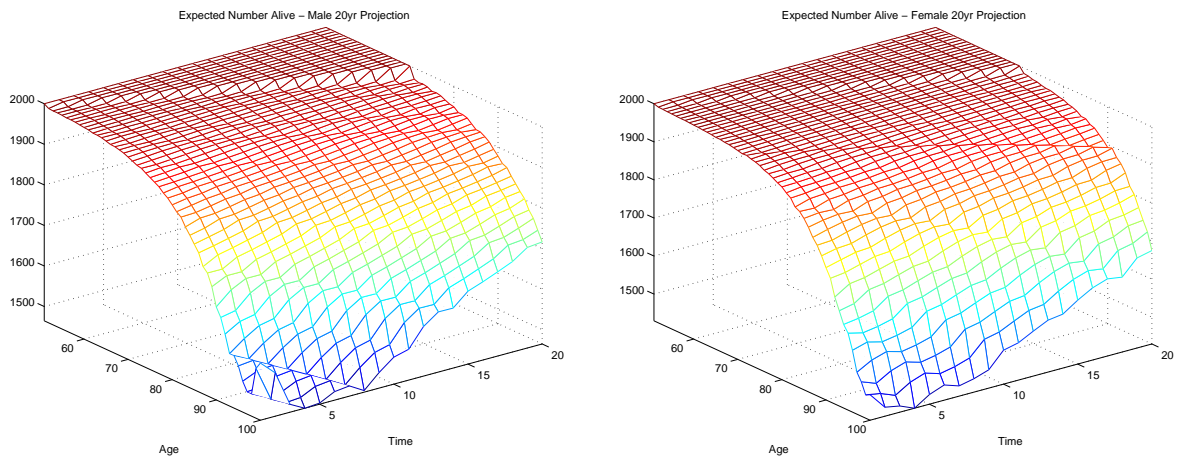

Figure 8: Expected number alive given $l(x, 0) = 2000$.

5 Conclusion

This paper develops a dynamic mortality model based on Australian data for use in assessing longevity risk and financial pricing of mortality linked securities. It follows the approach of Dahl (2004) [14] and Schrager (2006) [26] and models the development of mortality rates for multiple ages simultaneously. Most mortality models have restricted attention to modeling mortality at a single-age. The model developed in this paper considers mortality as a multivariate stochastic process over multiple ages. Maximum likelihood estimation is used to estimate trend and volatility parameters and principal componenets analysis to estimate dependence between ages.

The model provides a very good fit to Australian population mortality data for ages 50-99 from 1971-2004. The model is suitable for modeling insurance and annuity portfolios with a range of ages since the dependence between ages over time captures the risk of these cash flows. The model has its origins in financial models and this facilitates the application to the pricing of mortality and longevity-linked securities. Dependence between ages is an important component in the modelling of mortality. The use of principal components analysis allows the implementation of a model with a reduced number of factors driving mortality changes.

6 Acknowledgement

References

[1] Bauer, D., and Russ, J., (2006), Pricing Longevity Bonds using Implied Survival Probabilities. (Available at http://www.mortalityrisk.org/Papers/Models.html)

[2] Biffis, E., (2005), Affine Processes for Dynamic Mortality and Actuarial Valuations, *Insurance: Mathematics and Economics*, 37, 443-468.

[3] Booth, H., Maindonald, J., and Smith, L. (2002), Applying Lee-Carter under conditions of variable mortality decline. *Population Studies*, 56, 325-336.

[4] Booth, H., Hyndman, R. J., Tickle L., and De Jong, P., (2006), Lee-Carter Mortality Forecasting: A Multi-Country Comparison of Variants and Extensions, Working paper, Monash University.
(viewed at https://www.buseco.monash.edu.au/depts/ebs/pubs/wpapers/2006/wp13-06.pdf on 21 May 2008)

[5] Brouhns, N., Denuit, M., and Vermunt J.K. (2002), A Poisson Log-Bilinear Regression Approach to the Construction of Projected Life Tables, *Insurance: Mathematics and Economics*, Vol. 31: 373-393.

[6] Cairns, A. J. G., (2004), Interest Rate Models: An Introduction, Princeton University Press.

[7] Cairns, A.J.G., Blake, D., and Dowd, K., (2006a), Pricing Death: Frameworks for the Valuation and Securitization of Mortality Risk, *ASTIN Bulletin*, 36, No 1, 79-120

[8] Cairns, A.J.G., Blake, D., and Dowd, K. (2006b), A Two-Factor Model for Stochastic Mortality with Parameter Uncertainty: Theory and Calibration, *Journal of Risk and Insurance*, Vol. 73 (4): 687-718.

[9] Cairns, A.J.G., Blake, D., Dowd, K., Coughlan, G. D., Epstein, D., Ong, A. and Balevich, I., (2007), A Quantitative Comparison of Stochastic Mortality Models using data from England and Wales and the United States, Pensions Institute Discussion Paper PI 0701.

[10] Cox, J., Ingersoll, J., and Ross, S., (1985), A Theory of the Term-Structure of Interest Rates, *Econometrica*, 53, 385-408.

[11] Cox, S.H., and Lin, Y., (2007), Natural Hedging of Life and Annuity Mortality Risks, *North American Actuarial Journal*, 11 (3), 1-15.

[12] Currie, I. D., Durban, M. and Eilers, P. H. C., (2004), Smoothing and Forecasting Mortality Rates, Statistical Modelling, 4, 279-298.

[13] Denuit, M., Devolder, P., and Goderniaux, A.-C., (2007), Securitization of Longevity Risk: Pricing Survivor Bonds with Wang Transform in the Lee-Carter Framework, *Journal of Risk and Insurance*, 74(1), 87-113.

[14] Dahl, M., (2004), Stochastic Mortality in Life Insurance: Market Reserves and Mortality-Linked Insurance Contracts, *Insurance: Mathematics and Economics*, 35, 113-136.

[15] Dahl, M., and Møller, T. (2006), Valuation and Hedging of Life Insurance Liabilities with Systematic Mortality Risk, *Insurance: Mathematics and Economics*, Volume 39, Issue 2, 1, 193-217.

[16] Hyndman R. J. and Booth, H., (2006), Stochastic Population Forecasts using Functional Data Models for Mortality, Fertility and Migration, Working Paper, Monash University.

[17] Hyndman R.J. and Ullah, M. S., (2007), Robust forecasting of mortality and fertility rates: a functional data approach, *Computational Statistics & Data Analysis*, 51, 4942-4956.

[18] JP Morgan Pensions Advisory Group, (2007), LifeMetrics: A Tool for Measuring and Managing Longevity and Mortality Risks, Technical Document Version 1.0.

[19] Lee, R.D., and Carter, L.R., (1992), Modeling and Forecasting U.S. Mortality, *Journal of the American Statistical Association*, 87, 659-675.

[20] Liao, H. H., Yang, S. S. and Huang, I. H., (2007), The Design of Securitization for Longevity Risk: Pricing under Stochastic Mortality Model with Tranche Technique, Paper Presented to the 11th APRIA Conference, Taipei.

[21] Lin, Y., and Cox, S.H., (2005), Securitization of Mortality Risks in Life Annuities, *The Journal of Risk and Insurance*, 72 (2), 227-252.

[22] Macdonald, A. S., Bartlett, D., Berman, C., Daykin, C., Grimshaw, D. Savill, P. and Willets R., (2003), Mortality Improvements and the Cohort Effect, CMI Working Papers 1 and 2, Presented to the Staple Inn Actuarial Society on 11 March 2003, 43 pages (available at http://www.sias.org.uk/siaspapers/listofpapers/view_paper?id=ProjectingMortality viewed on 21 May 2008)

[23] Milevsky, M. A. and Promislow, S. D., (2001), Mortality Derivatives and the Option to Annuitise, *Insurance:Mathematics and Economics*, 29, 299-318.

[24] Renshaw, A.E., and Haberman, S. (2003), Lee-Carter Mortality Forecasting with Age-specific Enhancement, *Insurance: Mathematics and Economics*, Vol. 33: 255-272.

[25] Renshaw, A.E., and Haberman, S. (2006), A Cohort-Based Extension to the Lee-Carter Model for Mortality Reduction Factors, *Insurance: Mathematics and Economics*, Vol. 38: 556-570.

[26] Schrager, D. F., (2006), Affine Stochastic Mortality, *Insurance: Mathematics and Economics*, 38, 81-97.

[27] Sherris, M., and Wills, S., (2007), Financial Innovation and the Hedging of Longevity Risk, Presented at the Third International Longevity Risk and Capital Market Solutions Symposium, Taipei.

[28] Stallard. E. (2006), Demographic Issues in Longevity Risk Analysis, *Journal of Risk and Insurance*, Vol. 73 (4): 575-609

[29] Tuljapurkar, S. (1998), Forecasting Mortality Change: Questions and Assumptions, *North American Actuarial Journal*, Vol. 2, No. 4: 127-134.

[30] Tuljapurkar, S., and Boe, C. (1998), Mortality Change and Forecasting: How much and how little do we know?, *North American Actuarial Journal*, Vol. 2, No. 4: 13-47.

[31] Vasicek, O., (1977), An Equilibrium Characterisation of the Term Structure, *Journal of Financial Economics*, 5, 177-188.

[32] Wang, S.S., (2002), A Universal Framework for Pricing Financial and Insurance Risks, *ASTIN Bulletin*, 32, 213-234.

[33] Wills, S. and Sherris, M., (2008), Securitization, Structuring and Pricing of Longevity Risk, Working paper, School of Actuarial Studies, UNSW.

[34] Woodbury, M. A., and Manton, K. G. (1977), A Random-Walk Model of Human Mortality and Aging, *Theoretical Population Biology*, Vol. 11: 37-48.

[35] Yashin, A.I., Manton, K.G., and Vaupel, J. W. (1985), Mortality and Aging in a Heterogeneous Population: A Stochastic Process Model with Observed and Unobserved Variables, *Theoretical Population Biology*, Vol. 27: 154-175.