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Optimal pension asset allocation strategy for defined-contribution plans with exponential utility

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Abstract

This paper considers the asset allocation strategies for members of defined-contribution pension plans with exponential utility when there are three types of asset, cash, bonds and stocks. The portfolio problem is to maximize the expected utility of terminal wealth that uses the plan member's final wage as a numeraire, in the presence of three risk sources, interest risk, asset risk and wage risk. The use of a stochastic numeraire makes usual riskless cash assets risky. A closed form solution is found for the asset allocation problem when a portfolio replicates exactly the wage process exists, which is the true riskless asset. The optimal portfolio composition is horizon dependent, while the investments in the three asset classes have constant wealth-to-wage ratios. The paper discusses the implication of using wage as numeraire and assuming exponential utility function in portfolio and pension investment strategy studies.

Keywords: Defined-contribution pension plan; Wage risk; Optimal asset allocation; Exponential utility; Hamilton-Jacobi-Bellman equation.

1. Introduction

In recent years defined-contribution (DC) pension plans become more popular with corporate sponsors. Unlike the defined-benefits (DB) plans whose associated financial risks are borne by the plan sponsors, the members of a DC pension plan have to bear the financial risks. Several studies have investigated the optimal asset allocation strategies for DC pension plans (Battocchio and Menoncin 2004; Blake et al 2001; Boulier et al, 2001; Cairns et al, 2006; Deelstra et al, 2000, 2003). The optimal asset allocation strategy for a DC pension plan depends critically on the specifications of financial market, income model and terminal utility function. Difference in specifications can lead to substantially different conclusions.

The formulation of a financial market includes three aspects: 1) the process governing the instantaneous interest rate; 2) the number and types of asset for investment; and 3) the process governing the value of risky assets. In Merton's original portfolio and consumption strategy study (1969, 1971), interest rates and risk premiums are assumed to be constant. The assumption of a constant instantaneous interest rate in Merton's model is to simplify the computational complexity. It is obvious from empirical studies that stochastic variations in interest rates and in risk premiums exist, and it may not be appropriate to assume a constant instantaneous interest rate in portfolios with a long horizon such as pension funds. Later studies on consumption and portfolio strategies often use stochastic interest rates (Sorensen, 1999; Liu, 2001; Campbell and Viceira, 2002) or stochastic risk premiums (Kim and Omberg, 1996; Wachter 2002). Most pension studies assume a stochastic interest rate (Boulier et al 2001; Vigna and Haberman 2001; Haberman and Vigna 2002; Deelstra et al 2003; Cairns et al 2006; Battochio and Menoncin 2004), while constant interest rate scenarios may also be included in the same study (Cairns et al 2006).

As in Merton type consumption and portfolio work, most pension studies used two or three types of asset. Boulier et al (2001), Deelstra et al (2003) and Battochio and Menoncin (2004) used three assets: cash, bonds, and stocks. The stock price follows a geometric Brownian motion which includes volatilities from risk sources of both the interest rate and the stock market. Vigna and Haberman (2001) assumed two assets: one low risk asset and one high risk asset. Haberman and Vigna later (2002) extended their assumption into a sequence of N assets with increasing returns and volatilities. Cairns et al (2006) assume one riskless asset and n risky assets in the financial market.

Various treatments of wage incomes have been used in DC pension plan studies. In Boulier et al (2001), wages are continuously increasing at a constant rate and a pension plan member puts a constant part of her wage into the pension fund. Deelstra et al (2003) treat the contribution (a constant part of wage incomes) as a square integrable deterministic process. Haberman and colleagues assume a constant wage which can be considered equal to 1 for simplicity (Vigna and Haberman 2001; Haberman and Vigna 2002). Cairns et al (2006) and Battochio and Menoncin (2004) use stochastic wage incomes, which are governed by geometric Brownian processes. The wage income process includes volatilities from risk sources of the interest rate and the stock market, with or without a non-hedgeable volatility whose risk source is independent of financial market risk sources. Pension plan members will put a constant fraction of their wages into the pension fund.

The specification of the argument in a terminal utility function will substantially affect the asset allocation strategy for a pension plan. It is usually thought to be inadequate for pension plans to define the terminal utility as a function of the terminal pension wealth. Boulier et al (2001) and Deelstra et al (2000, 2003) have considered pension plans with a guaranteed minimum benefit at retirement and the terminal utility measured as a function of surplus cash over the guaranteed benefit. Battocchio and Menoncin (2004) use inflation as the pension wealth performance benchmark and thus suppose that the terminal utility is a function of real pension wealth (nominal wealth-to-price index ratio). Although these additional guarantee or benchmark may look innocent, the optimal allocation could be affected substantially (Samuelson 1989). While the performance of DC pension plans often uses the DB plan pension as benchmark, the pension from DB plans is usually based on final wages, implicitly assuming that pension income should be comparable to the existing standard of living. Such implicit assumption of the need to have income comparable to existing wage income in pension plans suggests some role of habit formation (Spinnewyn 1981; Becker and Murphy 1988) in terminal utility, and therefore final wages serve as a numeraire. In order to take the existing standard of living into account, Cairns et al (2006) assume that terminal utility is a function of the pension wealth-to-wage ratio or replacement ratio (pension income-to-final wage ratio).

Most portfolio and DC pension plan studies have assumed a power function for terminal utility (Boulier et al, 2001; Cairns et al, 2006; Deelstra et al, 2000, 2003; Campbell and Viceira, 2002). With power utility, analytical solution cannot be derived when there is a non-hedgeable wage risk. Although power utility is generally assumed to be more consistent with empirical data, in the opinion of Henderson (2005), Bliss and Panigirtzoglou (2004) find evidence from option prices that exponential utility provides better representation of preferences than power utility. Henderson (2005) solves explicitly the optimal portfolio choice problem with stochastic wage income and constant instantaneous interest rate for individuals with exponential terminal utility. The advantage of exponential (terminal) utility is its better tractability, so that the portfolio problem with non-hedgeable wage income can still have a closed-form solution. Battocchio and Menoncin (2004) assume an exponential terminal utility together with stochastic interest rates and wage incomes in their DC plan studies and also solve explicitly the optimal asset allocation problem. They found that the absolute risk aversion coefficient δ determines whether the portfolio is more, or less, affected by time-dependent components: a small absolute value of δ leads to an optimal strategy which is practically constant through time.

The present study extends the work of Battocchio and Menoncin (2004) by assuming terminal utility as a function of wealth-to-wage ratio which takes the current standard of living into consideration (Cairns et al 2006). The representation of the present work is mathematically very similar to that of Battocchio and Menoncin (2004), since in their work real pension wealth is represented by wealth-to-price index ratio. Although the financial market and the wage process are same in the present study as in Battocchio and Menoncin (2004), the optimal asset allocation strategy from the present study is very different. The optimal portfolio composition in risky assets consists of two components: (i) a speculative component, proportional to both the portfolio Sharpe ratio and the inverse of the relative risk aversion index, and (ii) a hedging component depending on the state variable parameters. For exponential terminal utility, the optimal portfolio composition is horizon dependent and the optimal investments in the three assets have constant wealth-to-wage ratios. This paper also shows that the formulation of wealth process and numerical solution in Battocchio and Menoncin (2004) contain some particular features or errors, which may affect the relevance of their final results to pension management in real world.

This paper is organized as follows. Section 2 formulates the financial market, wage and pension wealth growth models. Section 3 presents and solves the optimal asset allocation problem using the approach of Battocchio and Menoncin (2004). Section 4 presents and solves the optimization problem with a wage replicating portfolio. Section 5 discusses the results and concludes.

2. The model

This section presents the assets available for investment and their dynamics, and wage process.

2.1. Financial market

This study considers a financial market with no arbitrage that is frictionless and continuously open. The specifications of the financial market are similar to those in Boulier et al (2001) and Deelstra et al (2003), and same as those in Battocchio and Menoncin (2004). There are three types of asset in the financial market: cash, bonds and equities. For simplicity, only one equity asset, a stock, is considered, which can represent the index of a stock market. The randomness in the financial market is described by two standard and independent Brownian motions $Z_r(t)$ and $Z_S(t)$ with $t \in [0,T]$, defined on a complete probability space (W, F, P) where P is the real world probability. The filtration $F = F(t)$ $\forall t \in [0,T]$ generated by the Brownian motions can be interpreted as the information set available to the investor at time t.

The instantaneous risk-free rate of interest $r(t)$ follows an Ornstein-Uhlenbeck process (Vasicek model)

$$dr(t) = a(b - r(t))dt + s_r dZ_r(t),$$

$$r(0) = r_0, \qquad (1)$$

where a and b are strictly positive constants, and s_r is the volatility of interest rate. The Ornstein-Uhlenbeck process possesses a stationary distribution. The instantaneous drift $a(b - r(t))$ represents a force that keeps pulling the process towards its long term mean b with magnitude proportional to the deviation of the process from the mean (mean-reverting). The stochastic element $Z_r(t)$ causes the process to fluctuate around the level β in an erratic, but continuous fashion (Vasicek, 1977).

Given the interest rate process, the price of zero-coupon bonds for any date of maturity t at time t, $B(t, t, r)$, is governed by the diffusion equation (Vasicek 1977; Boulier et al 2001; Deelstra 2003)

$$\frac{dB(t,t,r)}{B(t,t,r)} = (r(t) + b(t,t)s_r x)dt - b(t,t)s_r dZ_r(t),$$

$$B(t,t) = 1,$$

where x is the market price of interest rate risk assumed to be constant, and

$$b(t,t) = \frac{1 - e^{-a(t-t)}}{a}.$$

The price process of the riskless asset, $R(t,r)$, is given by

$$dR(t) = R(t)r(t)dt,$$

$$R(0) = R_0. \qquad (2)$$

The riskless asset can be considered as a cash fund, i.e. a bank account paying the instantaneous interest rate $r(t)$ without any default risk. The value of units in the cash fund at t is then

$$R(t) = R(0)\exp\left[\int_0^t r(s)ds\right]. \qquad (3)$$

There are zero-coupon bonds for any date of maturity, and a bond rolling over zero coupon bonds with constant maturity *K*. The price of the zero coupon bond with constant maturity *K* is denoted by $B_K(t, r)$ with

$$\frac{dB_K(t,r)}{B_K(t,r)} = [r(t) + b_K s_r x]dt - b_K s_r dZ_r(t)\,, \qquad (4)$$

where

$$b_K = \frac{1-e^{-aK}}{a} \quad .$$

The relationship between $B(t, t, r)$ and $B_K(t,r)$ through the riskless cash asset $R(t)$ (Boulier et al, 2001) is

$$\frac{dB(t,t,r)}{B(t,t,r)} = \left(1 - \frac{b(t,t)}{b_K}\right)\frac{dR(t)}{R(t)} + \frac{b(t,t)}{b_K}\frac{dB_K(t,r)}{B_K(t,r)}\,.$$

As Boulier et al (2001) pointed out, it is quite unrealistic to assume the existence of zero-coupon bonds with arbitrary maturity date. However, since the interest rate model is a one-factor model, only one zero-coupon bond at any time is needed to replicate the other ones. The above equation shows that other bonds can be obtained through a portfolio of the riskless asset and the “rolling bond”, and that the “rolling bond” can be obtained by a portfolio of one zero coupon bond and the cash asset.

The total return (that is, the value of a single premium investment in the stock with reinvestment of dividend income) on the risky asset, the stock, is denoted by $S(t)$ with

$$dS(t) = S(t)[m_S(r,t)dt + v_{rS} s_r dZ_r(t) + s_S dZ_S(t)],$$

$$S(0) = S_0\,, \qquad (5)$$

where

$$m_S(r,t) = r(t) + s x_S \qquad (6)$$

is the instantaneous percentage change in stock price per unit time. The volatility $s = \sqrt{v_{rS}^2 s_r^2 + s_S^2}$, the total stock instantaneous volatility, is assumed to be constant, and v_{rS} represents a volatility scale factor measuring how the interest rate volatility affects the stock volatility. The market price of stock risk, x_S, is assumed to be constant, and the risk premium on the stock is $m_S = s x_S$.

The market as assumed above has a diffusion matrix given by

$$\Sigma \equiv \begin{bmatrix} -b_K s_r & 0 \\ v_{rS} s_r & s_S \end{bmatrix}, \tag{7}$$

and I assume that s_r and s_S are different from zero and the diffusion matrix is invertible.

2.2. Wages and contribution

The plan member's wage, $Y(t)$, evolves according to the stochastic differential equation (SDE)

$$dY(t) = Y(t)\left[(m_Y(t) + r(t))dt + v_{rY} s_r dZ_r(t) + v_{SY} s_S dZ_S(t) + s_Y dZ_Y(t)\right],$$

$$Y(0) = Y_0, \tag{8}$$

where $m_Y(t)$ is a deterministic function of time, age and other individual characteristics such as education and occupations. Here the instantaneous mean change of wages is assumed to be the sum of short interest rate and a deterministic function. Similar assumptions on wage processes have been used by Battochio and Menoncin (2004) and Cairns et al (2006). $Z_Y(t)$ is a standard Brownian motion, independent of $Z_r(t)$ and $Z_S(t)$. Here s_Y is a non-hedgeable volatility whose risk source does not belong to the set of the financial market risk sources, and assumed to be constant. When $s_Y = 0$, the market is complete. Otherwise the market is incomplete. Further, v_{rY} and v_{SY} are volatility scaling factors measuring how interest rate volatility and stock volatility affect wage volatility, respectively.

The plan member is assumed to contribute a constant proportion p of her wages to the pension fund, since most DC plans have a relatively fixed contribution ratio.

3. The optimization problem without a wage replicating portfolio and solution for exponential utility

This section presents the portfolio optimization problem and solves the problem for exponential utility using the approach of Battocchio and Menoncin (2004).

3.1. The fund wealth and wealth-to-wage ratio

The value of the plan member's pension fund is denoted by $W(t)$, and the proportions of fund wealth invested in the riskless asset, bonds and stock $q_R(t)$, $q_B(t)$ and $q_S(t)$ satisfy:

$$q_R(t) + q_B(t) + q_S(t) = 1, \tag{9}$$

The change in the pension wealth (dW) at time t comes from two sources: the capital income (returns from investment of pension wealth) and the contribution from wage income (Y) at time t. The dynamics of $W(t)$ is therefore governed by the SDE:

$$\begin{aligned} dW(t) &= W(t)\left[q_R \frac{dR}{R} + q_B \frac{dB}{B} + q_S \frac{dS}{S}\right] + pY(t)dt \\ &= \{W(t)[q_R r + q_B(r + b_K s_r x) + q_S m_S] + pY(t)\}dt \\ &+ W(t)(-q_B b_K + q_S v_{rS}) s_r dZ_r \\ &+ W(t) q_S s_S dZ_S , \end{aligned} \tag{10}$$

where p is the proportion of wage contributed to the pension plan and $Y(t)$ is the wage income at time t. The above equation indicates that the change in pension fund wealth is due to both the return of the investments and the contribution from wages.

In the study by Battocchio and Menoncin (2004), the change in pension fund wealth was assumed to be (in my own notation)

$$dW(t) = W(t)\left[q_R \frac{dR}{R} + q_B \frac{dB}{B} + q_S \frac{dS}{S}\right] + pdY(t).$$

This equation appears to imply that changes in wages (rather than wage itself) lead to the growth of pension wealth and that contribution from a constant wage will not increase pension wealth. I think that the contribution from the wage $Y(t)$ per se should be included, and might be more important in the growth of pension wealth. In the present formulation, even a constant wage still contributes to the growth of pension wealth. In an infinitesimal time interval *dt*, *Y*(*t*) can be considered as constant. Integration $\int pY(t)dt$ catches all effects of wage changes on pension contribution.

Since it is likely that a pensioner's utility derived from a pension fund at the time of retirement depends on its ratio to the individual's final wage, rather than its absolute value, the plan member is assumed to retire at time T and have a terminal utility function, $U(W(T),Y(T))$. That is, her terminal utility depends on both her final wage and her pension wealth at retirement. As in Cairns et al (2006), the terminal utility is assumed to be a function of the terminal pension wealth-to-wage ratio, $X(T)=W(T)/Y(T)$. Thus terminal utility will be independent of the interest rate at time *T*, *r*(*T*),

$$U(X(T),r(T)) \equiv U(X(T)) .$$

Applying Itô 's lemma, the SDE governing the wealth-to-wage ratio is

$$dX(t)=\frac{1}{Y}dW-\frac{W}{Y^2}dY+\frac{W}{Y^3}(dY)^2-\frac{1}{Y^2}(dWdY) . \qquad (11)$$

By substituting the expression of *W*, *Y*, *dW* and *dY*, the SDE governing this pension wealth-to-wage ratio process is:

$$dX(t)=(q'MX+p)dt+(q'\Gamma' X)dZ \quad , \qquad (12)$$

where,

$$q'\equiv[q_R \quad q_B \quad q_S],$$

$$M\equiv\begin{bmatrix} -m_Y+v_{rY}{}^2 s_r{}^2+v_{SY}{}^2 s_S{}^2+s_Y{}^2 \\ b_K s_r x-m_Y+(v_{rY}+b_K)v_{rY}s_r{}^2+v_{SY}{}^2 s_S{}^2+s_Y{}^2 \\ m_S-m_Y+v_{rY}(v_{rY}-v_{rS})s_r{}^2+v_{SY}(v_{SY}-1)s_S{}^2+s_Y{}^2 \end{bmatrix},$$

$$\Gamma'\equiv\begin{bmatrix} -v_{rY}s_r & -v_{SY}s_S & -s_Y \\ -(b_K+v_{rY})s_r & -v_{SY}s_S & -s_Y \\ (v_{rS}-v_{rY})s_r & (1-v_{SY})s_S & -s_Y \end{bmatrix},$$

$$Z\equiv[Z_r \quad Z_S \quad Z_Y]' . \qquad (13)$$

The new diffusion matrix for the financial market is given by Γ. I assume that $(\Gamma'\Gamma)$ is invertible in all following sections. The objective of the pension fund manager is to choose a portfolio strategy that maximizes the expected value of a terminal utility function. The terminal utility function is a function of the wealth-to-wage ratio $X(t)$.

In the study of Battocchio and Menoncin (2004), the terminal utility is a function of real wealth (the nominal wealth-to-price index ratio). The role of price index p in their wealth-to-price index ratio is similar to that of wage in the present wealth-to-wage ratio. However, in their formulation, the volatilities of price index appear in both Γ' and another vector Λ. Their formulation (in my notations) is

$$dX(t) = (q'MX + u)dt + (q'\Gamma'X + \Lambda)dZ$$

The vector Λ includes both wage volatilities and inflation (price index) volatilities. The term u has contributions from deterministic drifts and variances of both wage and inflation (price index) processes. It appears that the contribution from the inflation (price index) process has been fully captured by M and Γ (Ma, 2007) and the contribution from wage incomes should be represented by $pYdt$ in equation (10). If this view is correct, the SDE governing the real wealth growth in Battocchio and Menoncin (2004) should be

$$dX(t) = (q'MX + u)dt + (q'\Gamma'X)dZ$$

$$u = \frac{p}{p}Y$$

The difference in the treatment of inflation and wage impacts on the wealth-to-price index (or wealth-to-wage) ratio SDE will lead to a difference in the solution of the optimization problem.

3.2. The optimization problem and Hamilton-Jacobi-Bellman equation

The expected terminal utility has the functional form

$$V(t,x,r,y;q) = E\left[U(X_q(T), r(T), Y(T)) \mid X(t) = x, r(t) = r, Y(t) = y\right] \quad (14)$$

where $X_q(t)$ is the path of $X(t)$ given the strategy q. The objective of the optimization problem is to find the maximum expected terminal utility of a plan member,

$$J(t,x,r,y) = \sup_{q} V(t,x,r,y;q) \quad (15)$$

and the strategy q which attains this maximum. The above specifications have a similar form to those in Cairns et al (2006). The stochastic optimal control problem is rewritten as follows:

$$\max_{q} E[U(X(T),T)],$$

subject to

$$d\begin{bmatrix} w \\ X \end{bmatrix} = \begin{bmatrix} m_w \\ q'MX + p \end{bmatrix} dt + \begin{bmatrix} \Omega' \\ q'\Gamma' X \end{bmatrix} dZ,$$

$$w(0) = w_0, X(0) = X_0, \forall 0 \le t \le T, \quad (16)$$

where,

$$\underset{2\times 1}{w} \equiv [r \quad Y]',$$

$$\underset{2\times 1}{m_w} \equiv [a(b-r) \quad Y(m_Y + r)]',$$

$$\underset{2\times 3}{\Omega'} \equiv \begin{bmatrix} s_r & 0 & 0 \\ Yv_{rY}s_r & Yv_{SY}s_S & Ys_Y \end{bmatrix}. \quad (17)$$

The solution to this problem should give us the optimal portfolio composition.

The Hamiltonian corresponding to the above optimal control problem (16) is

$$H(J) = \frac{\partial J}{\partial t} + m'_w \frac{\partial J}{\partial w} + (q'MX + p)\frac{\partial J}{\partial X} + \frac{1}{2} tr\left(\Omega'\Omega \frac{\partial^2 J}{\partial w^2}\right) + (q'\Gamma' X)\Omega \frac{\partial^2 J}{\partial w \partial X}$$
$$+ \frac{1}{2}(q'\Gamma'\Gamma q X^2)\frac{\partial^2 J}{\partial X^2},$$

(18)

where $J(X,w,t)$ solves the Hamilton-Jacobi-Bellman equation and satisfies

$$J(X,w,t) = \sup_{\{q\}} E_t[U(X(T),T)$$

(Øksendal 2000). The two variables r and Y in equation (15) are represented by the vector $\underset{2\times 1}{w} \equiv [r \quad Y]'$ as defined in equations (17).

The system of the first order conditions on H with respect to q is:

$$\frac{\partial H}{\partial q} = MX \frac{\partial J}{\partial X} + \Gamma' \Omega X \frac{\partial^2 J}{\partial w \partial X} + (\Gamma' \Gamma q X^2) \frac{\partial^2 J}{\partial X^2} = 0 , \tag{19}$$

where $\frac{\partial H}{\partial q}$ is a vector. From the above equation, the optimal portfolio composition is:

$$q^* = -(\Gamma' \Gamma)^{-1} M \frac{J_X}{XJ_{XX}} - (\Gamma' \Gamma)^{-1} \Gamma' \Omega \frac{J_{wX}}{XJ_{XX}} , \tag{20}$$

where the subscripts on J indicate partial derivatives. Here $q^* = [q_R(t)^* \quad q_B(t)^* \quad q_S(t)^*]'$, the optimal proportions invested in cash, bonds and stock respectively.

The two terms on the right hand side of equation (20) can be designated as q_1* and q_2* respectively, which are themselves vectors with three elements corresponding to certain proportions of investment in cash, bonds and stock. We can also view q_1* and q_2* as two mutual funds constructed with three assets of cash, bonds, and stock, and obtain

Proposition 1: *Under the market structure assumed in this paper, the portfolio composition of the three assets (cash fund, bonds and stock) maximizing the investor's expected terminal utility (and pension wealth-to-final wage ratio) depends on two components: (i) a speculative component, proportional to both the portfolio Sharpe ratio and the inverse of the relative risk aversion index,*

$$q^*_1 = -(\Gamma' \Gamma)^{-1} M \frac{J_X}{XJ_{XX}} \tag{21}$$

and, (ii) a hedging component depending on the state variable parameters

$$q^*_2 = -\Gamma^{-1} \Omega \frac{J_{wX}}{XJ_{XX}} . \tag{22}$$

The first portfolio component increases when the "returns" on wealth-to-wage ratio $X(t)$ (i.e. M) increase, and decreases when the relative risk aversion ($-XJ_{XX}/J_X$) or the wealth-to-wage ratio variance ($\Gamma'\Gamma$) increases. Here the "returns" on wealth-to-wage ratio $X(t)$ (i.e. M) means wage adjusted returns on the assets, not the original returns from the assets.

The second portfolio component depends explicitly on the diffusion terms of the state variables (Ω), suggesting that this component covers the plan member from financial market risk. In fact, the present formulation uses the member's wage as a numeraire to assess the fund manager's performance.

This result is different from that of Battocchio and Menoncin (2004). In addition to these two components of optimal portfolios, Battocchio and Menoncin got a third preference-free hedging component depending only on the diffusion terms of assets and background variables with their formulation (2004). The preference-free hedging component in Battocchio and Menonion (2004) arises from the extra volatility term Λ in their wealth-to-price index SDE. As I explained in the preceding section, the inclusion of Λ appears incorrect (Ma 2007).

3.3. Optimal asset allocation strategy without a wage replicating portfolio for exponential terminal utility

When the instantaneous interest rate is constant and there is no wage income, individuals with exponential utility should invest a constant dollar value of wealth in risky assets and the rest in riskless assets (Merton 1969). The optimal allocation problem for exponential utility is solved in this section. By assuming an exponential utility function of the form

$$U(F) = -he^{-dF} \, . \qquad (23)$$

where h and d are strictly positive parameters and F is real pension wealth (nominal wealth-to-price index ratio), Battocchio and Menoncin find a closed-form solution for optimal pension management problem under stochastic interest rates, wages and inflation (2004). In this section, a similar exponential utility function as that of Battocchio and Menoncin (2004) is used. The terminal utility function is separable by product in wealth-to-wage ratio and in the other state variables according to the following form:

$$J(w,x,t) = -he^{-dX+h(w,t)},$$

$$J(w,x,T) = -he^{-dX}. \qquad (24)$$

Then, the derivatives of the maximum expected terminal utility

$$J_t = -he^{-dX+h}h_t,$$

$$J_X = hde^{-dX+h},$$

$$J_{XX} = -hd^2e^{-dX+h},$$

$$J_w = -he^{-dX+h}h_w,$$

$$J_{ww} = -he^{-dX+h}h_{ww} - he^{-dX+h}h_w{}^2,$$

$$J_{wX} = hde^{-dX+h}h_w, \qquad (25)$$

With exponential utility function, the optimization problem has a closed form solution, as long as $[\Gamma'\Gamma]$ is invertible. Here, as in the previous sections, $[\Gamma'\Gamma]$ is assumed to be invertible.

Substituting the partial derivatives of the value function, which was given in (25), and the optimal value of q into the Hamiltonian derived in the section 3.2, leads to the following equation:

$$H^* = -he^{-dX+h}h_t - m'_w h_w he^{-dX+h} + dhe^{-dX+h}\left[p + \left((\Gamma'\Gamma)^{-1}\frac{M}{d} + \Gamma^{-1}\Omega\frac{h_w}{d}\right)'M\right]$$
$$-\frac{1}{2}tr[\Omega'\Omega(h_{ww}he^{-dX+h} + h_w{}^2he^{-dX+h})] + \left[\left((\Gamma'\Gamma)^{-1}\frac{M}{d} + \Gamma^{-1}\Omega\frac{h_w}{d}\right)'\Gamma'\right]\Omega dh_w he^{-dX+h}$$
$$-\frac{1}{2}d^2he^{-dX+h}\left(-(\Gamma'\Gamma)^{-1}\frac{M}{d} - \Gamma^{-1}\Omega\frac{h_w}{d}\right)'\Gamma'\Gamma\left(-(\Gamma'\Gamma)^{-1}\frac{M}{d} - \Gamma^{-1}\Omega\frac{h_w}{d}\right) = 0.$$

(26)

Here, M, Γ, Ω and π are defined as those in the sections 2 and 3. Since

$$J(w, X(T), T) = -he^{-dX(T)},$$

it implies that

$$h(T, w(T)) = 0. \qquad (27)$$

The HJB equation can be simplified as follows

$$h_t + (m'_w - M'\Gamma^{-1}\Omega)h_w - dp - \frac{1}{2}M'(\Gamma'\Gamma)^{-1}M + \frac{1}{2}tr(\Omega'\Omega h_{ww}) = 0,$$

$$h(w,T) = 0. \qquad (28)$$

Let

$$f(w,t) \equiv -dp - \frac{1}{2}M'(\Gamma'\Gamma)^{-1}M. \qquad (29)$$

The partial differential equation can be solved by using the Feynman-Kac formula (Øksendal 2000 ; Duffie 2001), and the functional form of *h*(*w*,*t*) is given by:

$$h(w,t) = E_t\left[\int_t^T f(\tilde{w}(s),s)ds\right]. \qquad (30)$$

where

$$d\tilde{w}_s = (m_{\tilde{w}} - \Omega'\Gamma'^{-1}M)ds + \Omega(\tilde{w}_s,s)'dZ,$$

$$\tilde{w}_t = w_t, \qquad (31)$$

$$f(w,t) \equiv -dp - \frac{1}{2}M'(\Gamma'\Gamma)^{-1}M.$$

Then, the optimal portfolio strategy is

$$q^* = \frac{1}{dX}(\Gamma'\Gamma)^{-1}M + \frac{1}{dX}\Gamma^{-1}\Omega\int_t^T \frac{\partial}{\partial w_t}E_t[f(\tilde{w}_s,s)]ds. \qquad (32)$$

Here d can be considered as the Arrow-Pratt risk aversion measure. Under the Feynman-Kac representation theorem, the state variable *w* is given by the solution to a stochastic differential equation which is different from the original one. Since the terms p and $M'(\Gamma'\Gamma)^{-1}M$ in the function *f*(*w*,*t*) do not depend on the state variables, their derivatives with respect to w_t are zero and the above equation becomes

$$q^* = \frac{1}{dX}(\Gamma'\Gamma)^{-1}M \qquad (33)$$

As I have indicated in the section 3.2, the two items on the right hand side can be designated as two mutual funds, funds 1 and 2 respectively, which are constructed with cash, bonds and stock. The above equation corresponds to the fund 1 because fund 2 equals zero.

It is easy to show that equation (33) leads to constant wealth-to-wage ratio invested in the three assets, cash, bond, and stock. Since the denominator in the right hand side of the equation contain a factor X, the optimal wealth (to wage ratio) invested in the three assets

$$Xq^* = \frac{1}{d}(\Gamma'\Gamma)^{-1}M$$

The matrix product of the right hand side is a vector with three elements, each element for one asset category. Since all the three elements are constant, the optimal wealth-to-wage in each asset to wage ratio must be constant. The optimal total pension wealth-to-wage ratio invested in the three assets is

$$1'Xq^* = \frac{1}{d}1'(\Gamma'\Gamma)^{-1}M$$

where **1** $\in \Re^{3\times 1}$ is a vector containing only ones. Therefore, the optimal total pension wealth-to-wage ratio invested in the three assets must be constant.

This result of optimal constant total pension wealth-to-wage ratio is implied by the study of Battocchio and Menoncin (2004), who found that optimal pension wealth consists of a constant component and a horizon dependent component. Their optimal total pension wealth-to-price index ratio (real pension wealth) is horizon dependent because of the need to hedge wage and inflation risks.

Since an increase in pension wealth will increase the pension wealth derived utility, it seems wrong to have a constant optimal total pension wealth here. Moreover, to have the "constant optimal total pension wealth" invested in the three assets at the beginning of the pension plan, it is necessary to short-sell "riskless assets"; after the "constant optimal pension wealth" has been acquired, the ensuing contributions from wage incomes have to be invested in some assets instead of being thrown away. The existence of short-sale at the beginning and the need to invest the ensuing contributions suggest that the investment in the three assets is only part of the total pension wealth and that a fourth asset is needed for early short-sale and later long position. Because the stochastic wage is the numeraire, all the three assets, cash, bond and stock, are risky. The optimal strategy would be to hold some riskless assets as well as the constant wealth-to-wage ratio in the three risky assets. Since the stochastic wage is the numeraire, a portfolio that replicates the wage income is a riskless asset.

4. The optimization problem with a wage replicating portfolio and solution for exponential utility

This section presents the portfolio optimization problem and solves the problem for exponential utility when there is a wage replicating portfolio.

4.1. Optimization problem for non-hedgeable wage risk

If a portfolio P can be constructed to replicate the wage perfectly

$$dP(t) = P(t)\left[(m_Y(t) + r(t))dt + v_{rY} s_r dZ_r(t) + v_{SY} s_S dZ_S(t) + s_Y dZ_Y(t)\right], \tag{34}$$

the pension plan member now has four assets instead of three to invest. Let the proportion invested in the riskless portfolio be q_P,

$$q_P + q_R + q_B + q_S = 1. \tag{35}$$

The dynamics of $W(t)$ is now governed by the SDE:

$$\begin{aligned} dW(t) &= W(t)\left[(1 - q_R - q_B - q_S)\frac{dP}{P} + q_R\frac{dR}{R} + q_B\frac{dB}{B} + q_S\frac{dS}{S}\right] + pY(t)dt \\ &= \{W(t)[m_Y + r - q_R m_Y - q_B(m_Y - b_K s_r x) - q_S(m_Y - m_S)] + pY(t)\}dt \\ &+ W(t)[v_{rY} - q_R v_{rY} - q_B(v_{rY} + b_K) - q_S(v_{rY} - v_{rS})]s_r dZ_r \\ &+ W(t)[v_{SY} - q_R v_{SY} - q_B v_{SY} - q_S(v_{SY} - 1)]s_S dZ_S \\ &+ W(t)(1 - q_R - q_B - q_S)s_Y dZ_Y. \end{aligned} \tag{36}$$

Applying Itô's lemma, we get the wealth-to wage ratio expression that has exactly the same form as that of equations (12) and (13). The only difference is that the sum of proportions q_R, q_B and q_S is no longer 1, $q_R + q_B + q_S = 1 - q_P$. The HJB equation and optimal portfolio composition also have the same forms as equations (18) and (20) respectively. The solution for the exponential terminal utility has the same expressions as those from equation (26) to equation (33). Equation (33), $q^* = \frac{1}{dX}(\Gamma'\Gamma)^{-1}M$, shows that the because the optimal wealth-to-wage ratios invested cash, bond and stock are constant, the proportions they represent in the total pension wealth are decreasing as the total pension wealth is increasing. The proportion invested in the "riskless" replicating portfolio increases as the total pension wealth increases. This result is consistent with Merton's results (1969) that the optimal strategy for exponential utility is to invest a constant dollar value in the risky asset and the rest in the riskless asset.

From the above analysis, we get the following

Proposition 2: *When terminal utility is a function of wealth-to-wage ratio, under the market structure and optimization objective specified in this paper, the optimal pension asset allocation strategy is to hold a risky portfolio with constant (optimal) wealth-to-wage ratios in the three assets, cash, bond, and stock, and invest the rest of pension wealth in a riskless portfolio that replicates the wage.*

The constant optimal wealth-to-wage ratios invested in cash, bond and stock are

$$\begin{bmatrix} Xq_R\,* \\ Xq_B\,* \\ Xq_S\,* \end{bmatrix} = \frac{1}{d}(\Gamma'\Gamma)^{-1}M = \frac{1}{b_K s_r^{\,2} s_S^{\,2} s_Y^{\,2} d}\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} . \qquad (37)$$

The optimal proportions invested in cash, bond and stock are

$$\begin{bmatrix} q_R\,* \\ q_B\,* \\ q_S\,* \end{bmatrix} = \frac{1}{dX}(\Gamma'\Gamma)^{-1}M = \frac{1}{b_K s_r^{\,2} s_S^{\,2} s_Y^{\,2} dX}\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} , \qquad (38)$$

where

$$\begin{aligned} f_1 &= -m_Y(b_K - b_K v_{SY} - v_{rS}v_{SY} + v_{rY}) s_r^{\,2} s_S^{\,2} \\ &- s_r x[v_{rS}(b_K + v_{rS}) s_r^{\,2} s_Y^{\,2} + s_S^{\,2} s_Y^{\,2} + (v_{rS}^{\,2} v_{SY}^{\,2} + b_K v_{rS} v_{SY}^{\,2} \\ &- 2v_{rS}v_{rY}v_{SY} - b_K v_{rS}v_{SY} - b_K v_{rY}v_{SY} + b_K v_{rY} + v_{rY}^{\,2}) s_r^{\,2} s_S^{\,2}] \\ &+ m_S[(b_K v_{SY} + v_{rY}v_{SY} - v_{rS}v_{SY}^{\,2} - b_K v_{SY}^{\,2}) s_r^{\,2} s_S^{\,2} - (b_K + v_{rS}) s_r^{\,2} s_Y^{\,2}] \end{aligned}$$

$$\begin{aligned} f_2 &= m_Y(-v_{rS}v_{SY} + v_{rY}) s_r^{\,2} s_S^{\,2} \\ &+ s_r x[v_{rS}^{\,2} s_r^{\,2} s_Y^{\,2} + s_S^{\,2} s_Y^{\,2} + (v_{rS}v_{SY}^{\,2} - 2v_{rS}v_{rY}v_{SY} + v_{rY}^{\,2}) s_r^{\,2} s_S^{\,2}] \\ &- m_S[(v_{rY}v_{SY} - v_{rS}v_{SY}^{\,2}) s_r^{\,2} s_S^{\,2} - v_{rS} s_r^{\,2} s_Y^{\,2}] \end{aligned}$$

$$f_3 = -b_K m_Y v_{SY} s_r^{\,2} s_S^{\,2} - b_K s_r x[(v_{rY} v_{SY} - v_{rS} v_{SY}^{\,2}) s_r^{\,2} s_S^{\,2} - v_{rS} s_r^{\,2} s_Y^{\,2}]$$
$$+ m_S b_K s_r^{\,2} (s_Y^{\,2} + v_{SY}^{\,2} s_S^{\,2}) \tag{39}$$

The optimal proportion invested in the replicating portfolio is

$$q_P{}^* = 1 - q_R{}^* - q_B{}^* - q_S{}^*. \tag{40}$$

Since the expressions of optimal proportions for the three assets in equation (38) all contain δX in their denominator, the absolute risk aversion coefficient δ has no impact on the optimal proportions of the three assets in the risky portfolio. From equations (37)-(40), we have the following

Proposition 3: *When terminal utility is an exponential function of wealth-to-wage ratio, the absolute risk aversion coefficient d does not affect the optimal proportions of the three assets in the risky portfolio; d affects the relative weights between the risky and the riskless portfolios in the optimal allocation strategy described in Proposition 2.*

Proposition 3 is the result of the mutual fund theorem of Tobin (1958).

If there is no short-sale constraint, the pension plan member will short-sell the "riskless" replicating portfolio in order to hold the constant optimal wealth-to-wage ratio in the three "risky" assets, and pay off the short-sale with future contributions. After the short-sale has been paid off, the future contributions and the returns from the risky assets that exceed the constant optimal wealth-to-wage ratio will be invested in the "riskless" replicating portfolio. If the returns from the risky assets cannot maintain the constant optimal wealth-to-wage ratio, some future contributions have to be used to make up the difference. If there is short-sale constraint, the pension plan member will invest all contributions in the three risky assets until the constant optimal wealth-to-wage ratio in the risky assets is reached; afterwards the contributions will be invested in the "riskless" portfolio unless the returns from the risky assets cannot maintain the constant optimal wealth-to-wage ratio.

One interesting question consequent on Merton's results (1969) is for exponential utility how to invest in two or more risky assets when there is no risk free asset? The use of replacement ratio and wealth-to-wage ratio by Cairns et al (2006) and the present study, as well as the use of wealth-to-price index ratio by Battocchio and Menoncin, has actually made all assets (including cash) risky. The present result demonstrates that while the optimal wealth-wage ratio invested in the risky assets is constant, the rest of pension wealth should be invested in a “riskless” replicating portfolio.

Admittedly, a riskless portfolio may not always be found. With the assumption in this section of unhedgeable wage risk $Z_Y(t)$, the perfect replicating portfolio cannot be constructed with existing assets. In this case, the optimal pension allocation strategy is to invest the constant (optimal) wealth-to-wage ratios in the three assets, and the rest of pension wealth in a replicating portfolio that is closest to risk free.

Fig.1 shows the constant optimal wealth-to-wage ratio invested in the three (risky) assets, cash, bond, and stock, for different values of absolute risk aversion calculated with parameters in Table 1. The smaller the absolute risk aversion (ARA), the larger the optimal wealth-to-wage ratio invested in the three (risky) assets. Plausible optimal wealth-to-wage ratios invested in the three (risky) assets should not exceed the total pension wealth-to-wage ratios at retirement. The average (equivalent terminal) pension wealth-to-gross wage ratio of public pensions in the OECD countries is 9.4 for men (UK has the lowest at 4.6 and Greece the highest at 14.3, corresponding to a replacement ratio of 30.8% and 95.7% respectively) and 10.9 for women (Mexico has the lowest at 4.8, UK the second lowest at 5.3 and Greece the highest at 16.6) (Quesser and Whitehouse 2007). When ARA=1000 (logARA=3), the optimal pension wealth-to-wage ratio invested in the three (risky) assets is less than 0.5. Fig.2 shows the optimal wealth-to-pension ratio invested in the three (risky) assets for absolute risk aversion δ in the range between 30 (logARA=1.5) and 300 (logARA=2.5). In this range, the wealth-to-wage ratio invested in the three (risky) assets appears to be more realistic. For logARA=1.5, the optimal pension wealth-to-wage ratio is 15.1 for $v_{rS} = -1$ and 13.7 for $v_{rS} = 1$. For logARA=2.5, the optimal pension wealth-to-wage ratio is 1.51 for $v_{rS} = -1$ and 1.37 for $v_{rS} = 1$.

Table 1 Parameters used in numerical simulation

Interest rate	Value
Mean reversion, a,	0.2
Mean rate, b	0.05
Volatility, s_r	0.02
Initial rate, r_0	0.05
Fixed maturity bond	
Maturity, K	20 years
Market price of risk, x	0.15
Stock	
Risk Premium, m_S	0.06
Stock own volatility, s_S	0.19
Interest volatility scale factor, v_{rS}	1 or -1
Wage	
Wage premium, m_Y	0.01
Non-hedgeable volatility, s_Y	0.01
Interest volatility scale factor, v_{rY}	0.7
Stock volatility scale factor, v_{SY}	0.9
Initial wage, Y_0	10k
Contribution rate, p	10%
Length of pension plan, T	45

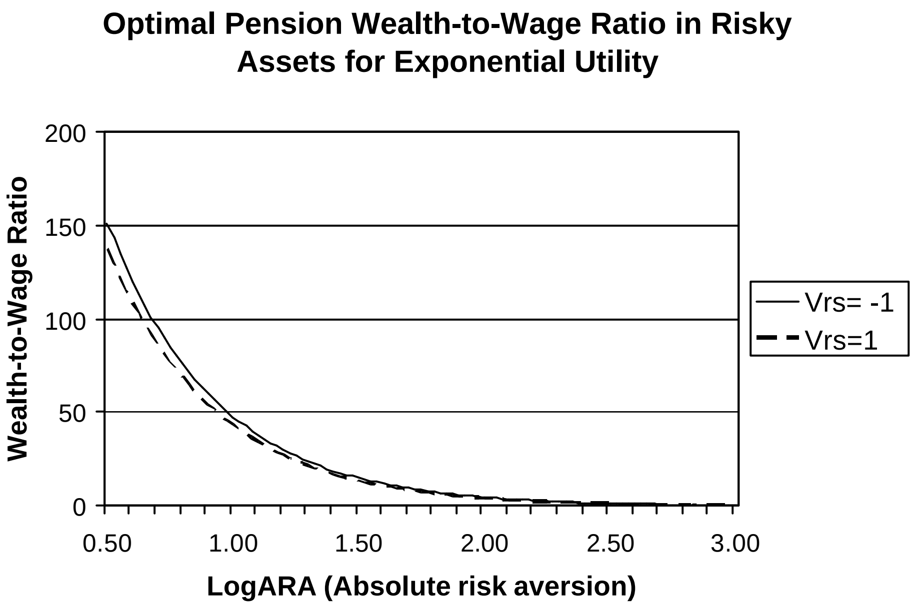


Fig.1 Relationship between absolute risk aversion (ARA) and optimal wealth-to-wage ratio of the three "risky" assets.

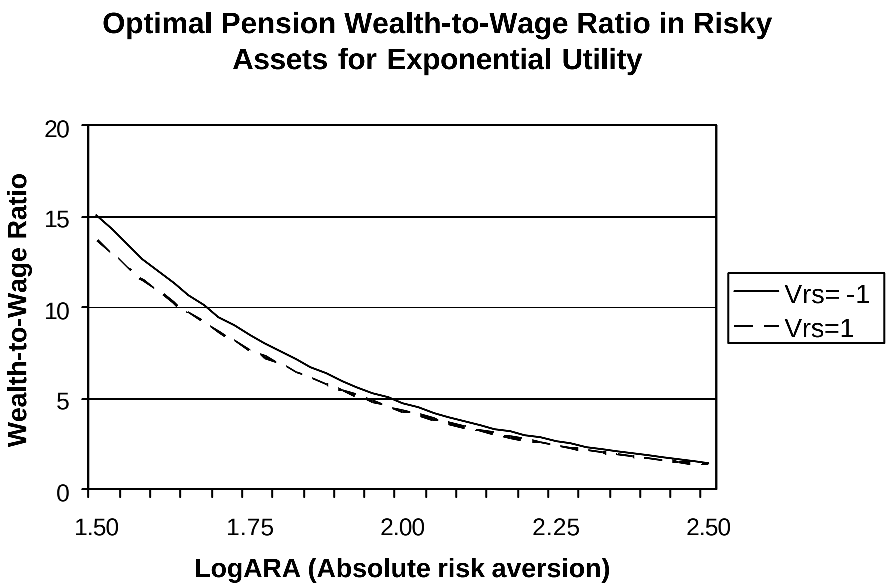


Fig.2 Relationship between absolute risk aversion (ARA) and optimal wealth-to-wage ratio of the three "risky" assets for ARA between 30 (logARA=1.5) and 300 (logARA=2.5).

Although the relative risk aversion has a generally agreed estimate range of 2-4, there seems to be no general agreement on the value of absolute risk aversion in literature. Here I try to use the relationship between relative risk aversion R(X) and absolute risk aversion A(X), $R(X) = XA(X)$, to estimate the values involved. Assuming that individuals with exponential utility have a relative risk aversion γ in the usual estimated range of 2 to 4 in terms of the wealth-to-wage ratio in the initial period, an assumption of δ=20 to 40 would be reasonable (in the initial period, the wealth-to-wage ratio X=π=0.1, the relative risk aversion R(X)=γ divided by X=0.1). A δ of 40 leads to an optimal wealth-to-wage ratio in risky assets of about 10, which is very similar to the average terminal wealth-to-wage ratio of public pensions in OECD countries, 9.4 for men and 10.9 for women (Queisser and Whitehouse 2007). The value used by Battocchio and Menoncin (2004), δ=20 (in the initial period with the wealth-to-wage ratio X=π=0.1 corresponding to a relative risk aversion γ=2), produces an optimal wealth-to-wage ratio in risky assets of about 20. A terminal pension wealth of 10 to 20 times the wage, which corresponds to an absolute risk aversion of 20 to 40, is close to the range of average pension wealth-to-wage ratio in OECD countries, 4.6 to 16.6 (Queisser and Whitehouse 2007).

4.2. Optimal asset allocation with hedgeable wage income contribution

If the wage income is fully hedgeable, the replicating portfolio P follows the price process

$$dP(t) = P(t)\left[(m_Y(t) + r(t))dt + v_{rY} s_r dZ_r(t) + v_{SY} s_S dZ_S(t)\right]. \qquad (41)$$

The pension plan member has four assets to invest and

$$q_P + q_R + q_B + q_S = 1.$$

The dynamics of $W(t)$ is governed by the SDE:

$$\begin{aligned} dW(t) &= W(t)\left[(1 - q_R - q_B - q_S)\frac{dP}{P} + q_R\frac{dR}{R} + q_B\frac{dB}{B} + q_S\frac{dS}{S}\right] + pY(t)dt \\ &= \{W(t)[m_Y + r - q_R m_Y - q_B(m_Y - b_K s_r x) - q_S(m_Y - m_S)] + pY(t)\}dt \qquad (42) \\ &+ W(t)[v_{rY} - q_R v_{rY} - q_B(v_{rY} + b_K) - q_S(v_{rY} - v_{rS})]s_r dZ_r \\ &+ W(t)[v_{SY} - q_R v_{SY} - q_B v_{SY} - q_S(v_{SY} - 1)]s_S dZ_S. \end{aligned}$$

Applying Itô 's lemma, we get the wealth-to wage ratio process

$$dX(t) = (q' MX + p)dt + (q' \Gamma' X)dZ \quad , \tag{43}$$

where,

$$q' \equiv [q_R \quad q_B \quad q_S],$$

$$M \equiv \begin{bmatrix} -m_Y + v_{rY}{}^2 s_r{}^2 + v_{SY}{}^2 s_S{}^2 \\ b_K s_r x - m_Y + (v_{rY} + b_K) v_{rY} s_r{}^2 + v_{SY}{}^2 s_S{}^2 \\ m_S - m_Y + v_{rY}(v_{rY} - v_{rS}) s_r{}^2 + v_{SY}(v_{SY} - 1) s_S{}^2 \end{bmatrix},$$

$$\Gamma' \equiv \begin{bmatrix} -v_{rY} s_r & -v_{SY} s_S \\ -(b_K + v_{rY}) s_r & -v_{SY} s_S \\ (v_{rS} - v_{rY}) s_r & (1 - v_{SY}) s_S \end{bmatrix},$$

$$Z' \equiv [Z_r \quad Z_S]. \tag{44}$$

Except that the elements in the matrices and vectors are different, the solution for the fully hedgeable wage scenario is the same as that represented by equations from (26) to (33). The constant optimal wealth-to-wage ratios invested in cash, bond and stock are

$$\begin{bmatrix} Xq_R{}^* \\ Xq_B{}^* \\ Xq_S{}^* \end{bmatrix} = \frac{1}{d} (\Gamma' \Gamma)^{-1} M = \frac{1}{b_K d} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} . \tag{45}$$

The optimal proportions invested in cash, bond and stock are

$$\begin{bmatrix} q_R{}^* \\ q_B{}^* \\ q_S{}^* \end{bmatrix} = \frac{1}{dX} (\Gamma' \Gamma)^{-1} M = \frac{1}{b_K dX} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} , \tag{46}$$

where

$$\begin{aligned} f_1 &= -m_Y (b_K - b_K v_{SY} - v_{rS} v_{SY} + v_{rY}) \\ &- s_r x (v_{rS}{}^2 v_{SY}{}^2 + b_K v_{rS} v_{SY}{}^2 - 2 v_{rS} v_{rY} v_{SY} - b_K v_{rS} v_{SY} - b_K v_{rY} v_{SY} + b_K v_{rY} + v_{rY}{}^2) \\ &+ m_S (b_K v_{SY} + v_{rY} v_{SY} - v_{rS} v_{SY}{}^2 - b_K v_{SY}{}^2) \end{aligned}$$

$$f_2 = m_Y(-v_{rS}v_{SY} + v_{rY})$$
$$+ s_r x(v_{rS}v_{SY}^2 - 2v_{rS}v_{rY}v_{SY} + v_{rY}^2)$$
$$- m_S(v_{rY}v_{SY} - v_{rS}v_{SY}^2)$$

$$f_3 = -b_K m_Y v_{SY} - b_K s_r x(v_{rY}v_{SY} - v_{rS}v_{SY}^2)$$
$$+ m_S b_K v_{SY}^2 \tag{47}$$

The optimal proportion invested in the replicating portfolio is

$$q_P{}^* = 1 - q_R{}^* - q_B{}^* - q_S{}^*.$$

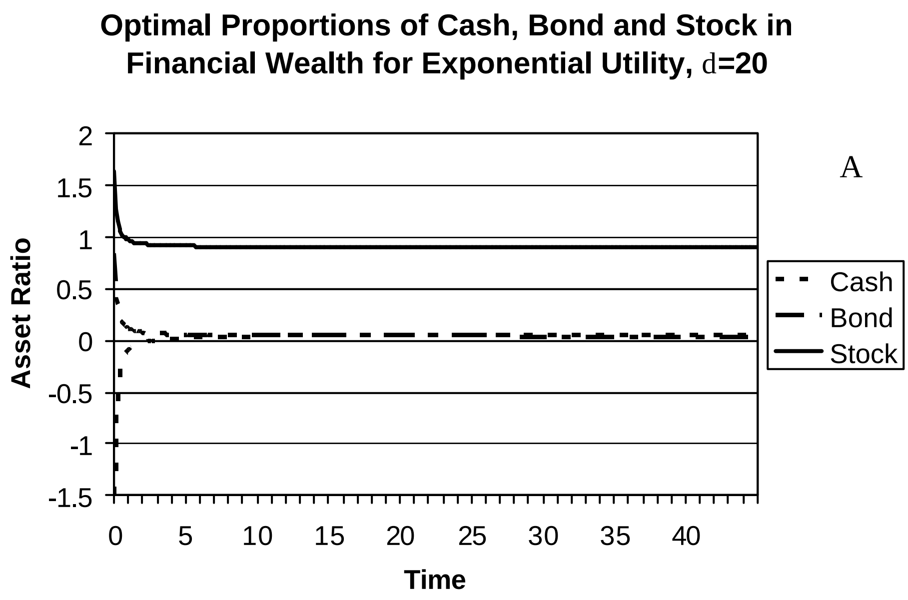
From the above analysis, we get

Proposition 4: *When there is no nonhedgeable wage risk, under the market structure and optimization objective specified in this paper, the optimal pension asset allocation strategy for exponential terminal utility is to invest constant (optimal) wealth-to-wage ratios in the three assets, cash, bond and stock, and the rest of pension wealth in a portfolio that perfectly replicates the wage process.*

The financial wealth of the plan members includes both the risky portfolio and the riskless portfolio. The riskless (wage replicating) portfolio can be constructed from the three assets, with the following proportions in cash, bond and stock

$$q_P = \begin{bmatrix} q_P{}^R \\ q_P{}^B \\ q_P{}^S \end{bmatrix} = \begin{bmatrix} 1 - v_{SY} - \dfrac{v_{rS}v_{SY} - v_{rY}}{b_K} \\ \dfrac{v_{rS}v_{SY} - v_{rY}}{b_K} \\ v_{SY} \end{bmatrix} \tag{48}$$

Fig.2 shows the numerical results on optimal proportions of cash, bond and stock in the financial wealth over time. At the beginning of the pension plan, the riskless portfolio is short-sold in order to hold the optimal amount of the optimal risky portfolio. As the short-sale is paid off gradually by wage contributions, the optimal proportions of the three assets in financial wealth stabilize. With a larger δ value, the optimal amount of the risky portfolio is smaller, hence less short-sale at the beginning of the pension plan.



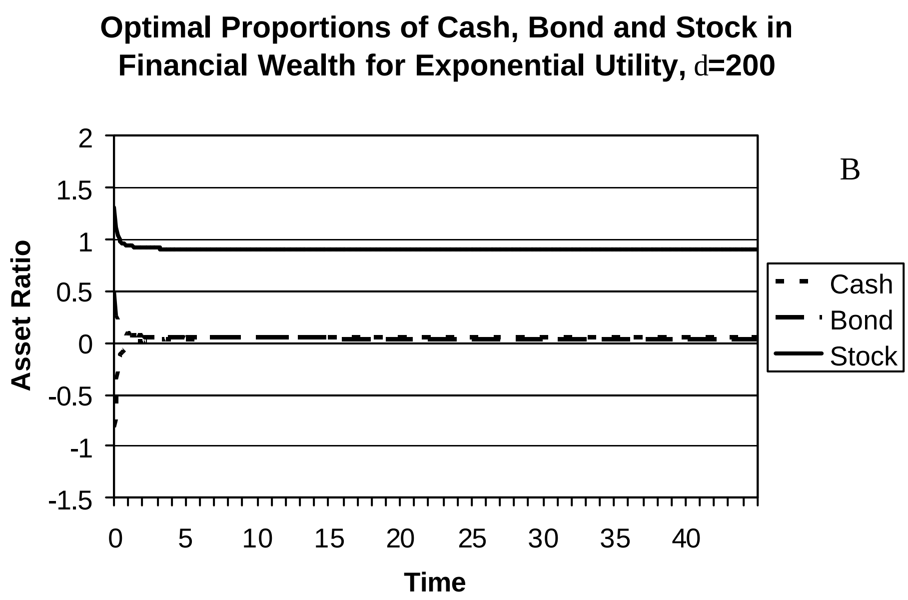
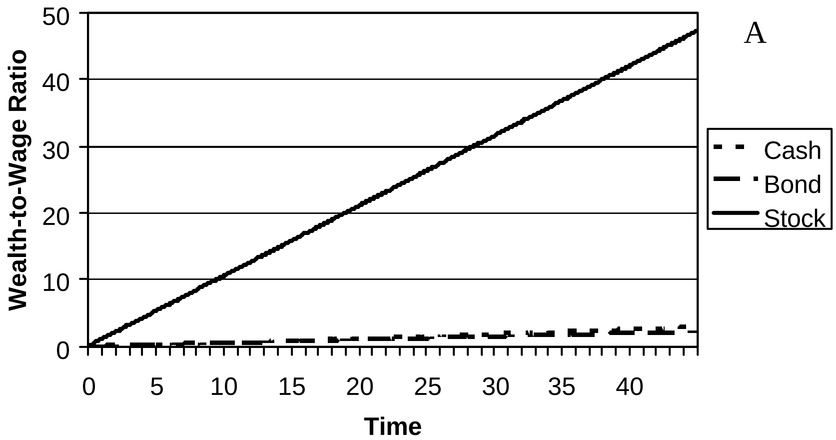


Fig.2 Optimal proportions of cash, bond and stock in financial wealth over the lifespan of pension plan. The parameters in Table 1 are used in the numerical simulation. The results are from 100 simulations. A. δ=20. B. δ=200.

Fig. 3 shows the numerical results on wealth-to-wage ratio invested in the three assets for the financial wealth of the pension plan. With assumptions on the financial market and wage process in the present paper as well as in Battocchio and Menoncin (2004), the largest proportion is invested in the stock. The larger the δ value, the smaller the expected terminal wealth-to-wage ratio.

Cash, Bond and Stock Financial Wealth to Wage Ratio for Exponential Utility, d=20



Cash, Bond and Stock Financial Wealth to Wage Ratio for Exponential Utility, d=200

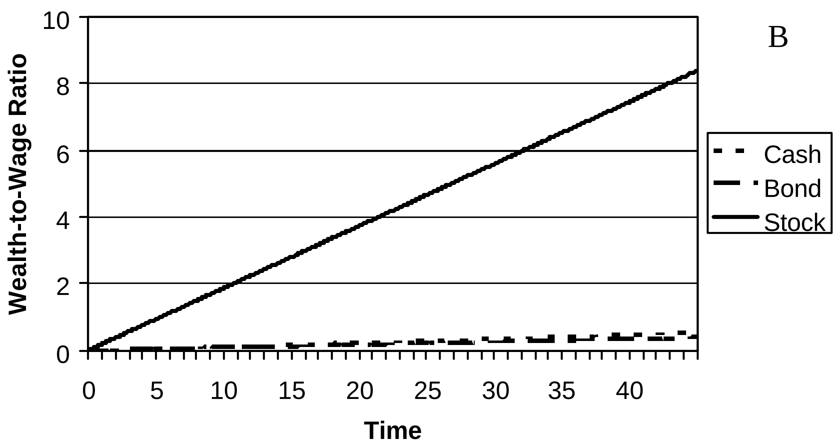


Fig.2 The optimal wealth-to-wage ratio invested in cash, bond and stock for financial wealth over the lifespan of pension plan. The parameters in Table 1 are used in the numerical simulation. The results are from 100 simulations. A. δ=20. B. δ=200.

The present results on optimal asset allocation strategy for exponential terminal utility indicate a stochastic lifestyle strategy due to the contributions from wage income, consistent with earlier studies by Bodie et al (1992), Campbell and Viceira (2002), Deestra et al (2003); Battocchio and Menoncin (2004), and Cairns et al (2006). The present study assumes the same asset return and wage processes as those in Battocchio and Menoncin (2004), but they failed to include the investment in a "riskless" portfolio, which in their case should replicate the price index. Their results correspond to my initial results in section 3 when the investment in the "riskless" portfolio is not included. Therefore, they might have made a mistake in treating the pension wealth invested in "risky" assets as the total pension wealth. This is shown in their numerical simulations. Battocchio and Menoncin did not specify the initial pension wealth value in their numerical simulations. Since the initial wage is 100 and the contribution rate 12%, it is very unlikely that the average realized pension wealth at t=0 or soon after can be as high as >23. Given that both the expected wage growth and the mean interest rate are higher than inflation growth, 12% wage incomes (with an initial wage of 100) invested in riskless assets over 40 years should generate a real wealth of at least 480. The numerical simulation by Battocchio and Menoncin, however, only generated a terminal real wealth of <30 after 40 years' contribution and investment, much lower than the initial one period wage of 100 (Battocchio and Menoncin 2004).

If we view the total real pension wealth of Battocchio and Menoncin (2004) as the real pension wealth invested in the three risky wealth, their optimal real wealth (wealth-to-price index ratio) in the risky assets is horizon dependent, whereas in the present study the optimal wealth-to-wage ratio is constant. The difference arises because of the need to hedge wage and inflation risks in their study. They find that the optimal real pension wealth (invested in the three "risky" assets) consists of a constant component and a horizon dependent component. In their numerical simulation, they use an absolute risk aversion coefficient δ=20 and an initial wage Y(0)=100, which implies a relative risk aversion coefficient γ=200 at least (with contribution rate π=10% of wage income). Battocchio and Menoncin noted that for smaller δ (with respect to initial wage Y(0)), the optimal strategy is practically constant through time (2004). The use of wealth-to-wage ratio and the model formulation in this paper removes the need to hedge wage and interest rate risks, leading to a constant optimal wealth-to-wage ratio that needs to be invested in the three risky assets.

Another difference between my present results and those of Battocchio and Menocin (2004) is that they found three components (preference-free hedging, speculative, and state variable dependent hedging) in the general expression of the optimal composition, whereas the present results only have two (speculative and state variable dependent hedging). As explained earlier, the preference-free hedging component may arise from their incorrect treatment of wage and inflation contributions to the real wealth (wealth-to-price index ratio). In the present study, the state variable dependent hedging component in the proportions invested in the three "risky" assets disappears because the use of wealth-to-wage ratio as formulated in this paper removes the need to hedge interest rate and wage risks. The model used by Battocchio and Menoncin (2004) still needs to hedge wage and inflation risk even when formulated correctly (that is, $u = \frac{p}{p} Y$ and Λ=0 in their real wealth growth SDE $dX(t) = (q' MX + u)dt + (q' \Gamma' X + \Lambda) dZ$). Therefore, they found that the value and portfolio composition of the optimal real wealth (wealth-to-price index ratio) invested in the three risky assets is time/horizon dependent, whereas the present study shows that the optimal value and portfolio composition of the optimal wealth-to-wage ratio invested in the three risky assets is horizon independent.

5. Conclusion

This paper has solved the optimal portfolio problem under stochastic interest rate and wage income for DC pension plan members with exponential utility, using three assets, cash, bonds and stock. The terminal utility of a pension plan holder is assumed to be a function of terminal pension wealth-to-wage ratio. The general results of the present paper differed from earlier findings (Battocchio and Menoncin 2004; Cairns et al 2006). The use of stochastic wages makes assets that do not co-vary perfectly with wage risky. Under the present model assumptions, it is found that the optimal composition in "risky" assets contains two components: a speculative component proportional to both portfolio Sharpe ratio and the inverse of the Arrow-Pratt relative risk aversion index, and a hedging component dependent on the state variable parameters.

When the expected terminal utility is an exponential function of wealth-to-wage ratio, a closed form solution is derived for the optimal asset allocation problem when a wage replicating portfolio exists. The hedging component dependent on the state variable parameters disappears and the optimal portfolio composition and the investments in the "risky" assets have constant optimal pension wealth-to-wage ratio. Using wealth-to-wage ratio, as the argument of the expected terminal utility, in the present study removes the dependence of instantaneous conditional expected change per unit time (the expression multiplying with dt in the SDE) on the state variables, so that the need to hedge against the fluctuations in the state variables disappears. The preference free hedging component in Battocchio and Menoncin (2004) seems to result from their incorrect treatment of wage income and price index in their real wealth growth process. It seems that excluding a true riskless asset (portfolio) and working directly with optimal wealth instead of optimal proportions also lead to incorrect numerical simulation results in Battocchio and Menoncin (2004).

To summarize, this paper derives a close form solution of the optimal asset allocation problem for DC pension plan members with exponential terminal utility that is a function of wealth-to-wage ratio, when a wage replicating portfolio exists; the optimal portfolio composition is horizon dependent while investments in cash, bonds and stocks have constant optimal wealth-to-wage ratio.

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