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under Interest-Rate and Longevity Risks

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Risks**

By

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Abstract

This paper examines the impact of interest-rate risk and longevity risk on the distribution of annuity prices in the distant future. To so, the paper uses a computationally efficient algorithm that simulates the state variables out to the end of the horizon period and then uses a Taylor series approximation to compute approximate annuity prices at the end of that period. Illustrative results suggest that annuity prices are likely to rise considerably, but are also quite uncertain. These findings have some unpleasant implications for future pensioners and those who will have to look after them.

JEL Classification Numbers: G23, C15, C53

Key words: longevity risk, interest-rate risk, annuity prices, Taylor series approximation, pension risk

I. INTRODUCTION

This paper examines the impact of interest-rate risk and longevity risk on the distribution of annuity prices in the distant future. At first sight, this might appear to be a rather arcane issue. Most people do not care much about current annuity prices, so why would they care about possible annuity prices in, say, 40 years' time? None the less, we would suggest that the future prices of annuities are much more important than they might first appear to be. One reason is suggested by the global increase in longevity: people are living longer than previously anticipated, and it is natural to ask how much this is likely to cost. To answer this question, we need some index of the cost of life expectancy, and a natural index is the cost of a life annuity, i.e., the expected present value of an income stream of \$1 per annum until the buyer of the annuity dies. The cost of increased longevity is then the likely increase in future annuity prices. Thus, an assessment of likely future annuity prices is a key to evaluating the likely cost of higher longevity.

There is a second and often more personal reason why we should care about future annuity prices. Consider the illustrative case of a male currently aged 25 who is starting a defined contribution (DC) pension plan or a 401k pension plan in the U. S. and is planning to retire in, say, 40 years' time at the age of 65. He anticipates that when he reaches that age, he will convert his accumulated pension fund into a life annuity in order to hedge his own longevity risk and avoid outliving his own financial resources. The value of his retirement income will depend not only on the value of his pension fund, but also on the price of annuities at that time. Other things being equal, this means that his retirement income prospects will be affected by the distribution of

future annuity prices: the greater the dispersion of that distribution, the more risky his retirement income will be. Hitherto, analyses of DC/401k plans have tended to focus on the risks facing the member that arise from the uncertainty of the future value of the pension fund itself, i.e., they have tended to focus on investment and contribution risks during the accumulation stage of the plan, and in so doing have tended to overlook the impact of distribution stage risks, most notably longevity risk, that can affect retirement income through their impact on annuity prices. The superficially arcane issue of the distribution of future annuity prices therefore turns out to be a key ingredient in determining the risks associated with DC pension plans.

This paper investigates this issue further and provides some illustrative results for a long-term (40-year) horizon. As is well-known, the principal factors that determine annuity prices are life-expectancy prospects (which depend on future mortality rates) and interest rates. To model future annuity prices, we therefore need a model of future mortality rates. We also need a model of future interest rates, since these will influence the future price of the bonds the annuity provider buys in order to make annuity payments and, in turn, the discount rate used to value the annuity itself. The models we use involve stochastic simulation, but any naïve attempt to use stochastic simulation runs into a problem: if we simulate the state variables out to some future horizon period T , we then face the problem of how to obtain the future time- T annuity prices contingent on the time- T values of the state variables. The most obvious approach is to use stochastic simulation for this purpose, but we would then find ourselves running ‘simulations within simulations’ which can be computationally very expensive. To operate within feasible real-time constraints, we need to find some alternative method to obtain the future annuity prices contingent on the outcomes of

the time- T state variables. We solve this problem using a Taylor series approximation: we simulate the state variables out to T , and then use the Taylor series approximation to estimate the annuity prices at T as functions of the values of the state variables.

This paper is organized as follows. Section 2 describes the stochastic mortality model used to model longevity risk, and section 3 outlines the stochastic interest-rate model that we use.¹ Section 4 explains computational issues and proposes a ‘Taylor series approximation within simulation’ approach that allows us to estimate future annuity prices using simulated future values of the underlying state variables. Section 5 presents some illustrative results, and Section 6 concludes and elaborates on the implications of our findings for the riskiness of DC plan retirement incomes.

II. A STOCHASTIC MORTALITY MODEL

We model mortality stochastically using an approach initially set out in Cairns *et alia* (2006) but using the parameterisation of Cairns *et alia* (2007). Let $q(t, x)$ be the realized mortality rate in year $t+1$ (that is, from time t to time $t+1$) for individuals aged x at time t . We assume that $q(t, x)$ is governed by a two-factor Perks stochastic process:

$$(1) \quad q(t, x) = \exp[\kappa_1(t+1) + \kappa_2(t+1)(x - \bar{x})] / \{1 + \exp[\kappa_1(t+1) + \kappa_2(t+1)(x - \bar{x})]\}$$

¹ The present paper assumes that longevity and interest risks are independent of each other. An exploration of the validity of this assumption is left for future research.

where $\kappa_1(t+1)$ and $\kappa_2(t+1)$ are themselves stochastic processes that are measurable at time $t+1$ (see Perks, 1932, Benjamin and Pollard, 1993), and \bar{x} is a constant that is typically set to the mean of the range of ages used to calibrate the model. Now let $\kappa(t) = (\kappa_1(t), \kappa_2(t))'$ and assume that $\kappa(t)$ is a random walk with drift:

$$(2) \quad \kappa(t+1) = \kappa(t) + \mu + CZ(t+1)$$

where μ is a constant 2×1 vector of drift parameters, C is a constant 2×2 lower triangular Choleski square root matrix of the covariance matrix V (that is, $V = CC^T$), and $Z(t)$ is a 2×1 vector of independent standard normal variables. Cairns *et alia* (2006, 2007) show that this model provides a good fit to UK Government Actuary's Department (GAD) data for English and Welsh males over 1961-2004.

Now let $S(t, x)$ be the survivor index at time t of a cohort aged x in year 0: that is, $S(t, x)$ is the probability, measured retrospectively, that an individual aged x at time 0 survives to time t . For any given x , $S(0, x) = 1$ and $S(t, x)$ will decrease as t gets bigger and eventually approach 0 as t gets large. Given any path of $q(t, x)$ as obtained above, we then obtain a corresponding path of $S(t, x)$ from the relationship between mortality and the survivor index:

$$(3) \quad S(t+1, x) = (1 - q(t, x+t))S(t, x)$$

Note that these survivor rates are driven off the state variables $\kappa_1(t)$ and $\kappa_2(t)$. For our purposes, we wish to simulate sets of state variables out to a future time T , and then estimate the expectations of (3) conditional on surviving to the specified future date and conditional on the future values of the state variables $\kappa_1(T)$ and $\kappa_2(T)$ at that date.

III. A STOCHASTIC INTEREST-RATE MODEL

We also need a model of the interest-rate process, and the simplest model that meets our requirements is the Cox-Ingersoll-Ross (CIR) model (1985). This model postulates that the instantaneous spot interest rate r obeys the following continuous-time process:

$$(4) \quad dr(t) = \alpha(\bar{r} - r(t))dt + \sigma\sqrt{r(t)}dW(t)$$

where α indicates the strength of the mean-reversion process governing r , \bar{r} is the mean instantaneous spot interest rate, σ is the interest-rate volatility and $dW(t)$ is a standard geometric Brownian motion. This model is attractive because it allows for interest rates to be mean-reverting but does not allow them to become negative. Another attractive feature is that it gives us a straightforward formula for the spot rate term structure based on the current instantaneous spot rate: if $R(t, T)$ is the time- t spot rate for the fixed maturity date T (that is, $T-t$ years to maturity), then

$$(5) \quad R(t, T) = -(T - t)^{-1} \ln P(t, T)$$

where $P(t, T)$ is the time- t price of a zero-coupon bond with maturity T and where²

$$P(t, T) = A(t, T) \exp[-B(t, T)r(T)]$$

$$A(t, T) = \left(2\gamma \exp[(\alpha + \gamma)(T - t) / 2] / \{(\gamma + \alpha)(\exp[\gamma(T - t)] - 1) + 2\gamma\} \right)^{2\bar{r}\alpha/\sigma^2}$$

$$B(t, T) = 2(\exp[\gamma(T - t)] - 1) / \{(\gamma + \alpha)(\exp[\gamma(T - t)] - 1) + 2\gamma\}$$

$$\gamma = \sqrt{\alpha^2 + 2\sigma^2}$$

From a computational perspective, the CIR model is appealing because the exact distribution of the instantaneous spot interest rate under the CIR model is known. To be precise, if $r(T)$ follows a CIR process, then $(4\alpha r(T)) / \{\sigma^2(\exp[\alpha T] - 1)\}$ has a non-central chi-squared distribution with $4\alpha\bar{r}/\sigma^2$ degrees of freedom and a non-centrality parameter equal to $(4\alpha r(0)) / \{\sigma^2(\exp[\alpha T] - 1)\}$ (Cairns, 2004, Theorem 4.8 (c)). This means that we can simulate values of $r(T)$ directly from their exact distribution using the CIR parameters and the current instantaneous spot rate $r(0)$ as inputs. Once we have these terminal instantaneous spot rate values, we can then use (4) to infer the spot-rate term structures contingent on each of these values.

A typical distribution for $r(T)$ is given in Figure 1 and Table 1, based on assumed values of $\alpha = 0.20$, $\sigma = 0.10$ and $\bar{r} = 0.04$. Figure 1 shows the pdf and

² For simplicity we have assumed here that there is no difference between the risk-neutral and the real world probability measures. However, where these probability measures are different, α , σ and \bar{r} in the price formulae must be interpreted as parameters under the risk-neutral measure.

Table 1 gives some of its key parameters. We can see that the pdf has a strong positive skew and a long right-hand tail. What is perhaps most striking about this pdf is the extent of its dispersion – for example, 30% of simulated values are under 0.02, 30% are above 0.049 and we get a small number of very high values in the long right-hand tail – and this is the case even though the spot-rate process is mean-reverting.³

INSERT FIGURE 1

INSERT TABLE 1

These findings suggest that we might expect interest-rate risk to have a considerable impact on the distribution of future annuity prices.

IV. COMPUTATIONAL ISSUES

If we combine these models, we have three random state variables: the $\kappa_1(\cdot)$ and $\kappa_2(\cdot)$ state variables from the mortality model, and the instantaneous spot interest rate from $r(\cdot)$ from the interest-rate model. The current time is time 0 and we would like to be able to simulate these state variables out to some future period T . Suppose, then, that we take j simulation paths of each state variable out to period T , and let $[\kappa_1^j(T), \kappa_2^j(T), r^j(T)]$ be the j^{th} set of simulated state variables for period T .

The annuity price at time T , $a(T)$, then depends on the values of these state

³ It is also worth noting that the CIR process is widely regarded by interest-rate practitioners as *under-estimating* the distribution of instantaneous spot rates at long horizons. To the extent that this is the case, then our later results will under-state the impact of interest-rate risk on the distribution of future annuity prices and therefore under-estimate the riskiness of DC pension plans.

variables, so we will, from time to time, use the extended notation $a(T) \equiv a(T; \kappa_1(T), \kappa_2(T), r(T))$ to reflect the price's dependence on the state variables. Similarly, $R(T, T+i) \equiv R(T, T+i; r(T))$. The annuity price conditional on the simulated state variables under simulation path j is thus

$$(6) \quad a^j(T) = a(T; \kappa_1^j(T), \kappa_2^j(T), r^j(T)) = (1 + \phi) \sum_{i=1}^{50} \exp(-iR(T, T+i; r^j(T))) E[S(T+i, x) / S(T, x) | \kappa_1^j(T), \kappa_2^j(T)]$$

where ϕ is the loading factor built into the annuity price and $R(T, T+i; r(T))$ is the spot interest rate prevailing over the period from T to $T+i$ given $r(T)$. Equation (6) is a sum of future expected survivor rates from age 66 to age 115 (conditional on surviving to age 65) multiplied by market zero-coupon bond prices, and we implicitly assume that no-one lives beyond age 115. The term $E[S(T+i, x) / S(T, x) | \kappa_1^j(T), \kappa_2^j(T)]$ is to be interpreted as the expected probability that an individual currently aged x will survive to year $T+i$ conditional on their surviving to T and conditional on the mortality state parameters at T .

However, we cannot compute (6) directly because there is no simple formula for the $E[S(T+i, x) / S(T, x) | \kappa_1(T), \kappa_2(T)]$ in terms of the mortality state variables $\kappa_1^j(T)$ and $\kappa_2^j(T)$. The most obvious solution to this problem would be to use stochastic simulation to estimate $E[S(T+i, x) / S(T, x) | \kappa_1(T), \kappa_2(T)]$, for each set of $\kappa_1^j(T)$ and $\kappa_2^j(T)$. For example, we might use m simulation trial paths to generate $j=1, \dots, m$ pairs of simulated time- T state variables $\kappa_1^j(T)$ and $\kappa_2^j(T)$.

From each such pair, we might then generate $k=1, \dots, m$ sets of paths for the post- T survivor rates $S^{jk}(T+i, x) / S^{jk}(T, x)$ and obtain $E[S^j(T+i, x) / S^j(T, x) | \kappa_1^j(T), \kappa_2^j(T)]$ as their average. In principle, this solution would work, but it involves us taking m^2 simulation paths and this would be computationally extremely expensive.⁴

We need to find some way of reducing this computational burden, and one way to do this is to use a Taylor series approximation. Accordingly, following Cairns (2007), let $\kappa = [\kappa_1(T), \kappa_2(T)]'$ and let $\hat{\kappa} = E[\kappa]$ be the expectation of the mortality state variables at T . We now define $f(i, x, \kappa) = \Phi^{-1}(E[S(T+i, x) / S(T, x) | \kappa = (\kappa_1(T), \kappa_2(T))])$ as the probit transformation of $E[S(T+i, x) / S(T, x) | K = (K_1(T), K_2(T))]$, where $\Phi(\cdot)$ is the standard normal distribution function.⁵ We then take the following second-order Taylor series expansion of $f(i, x, \kappa)$ around $\hat{\kappa}$:

$$(7) \quad f(i, x, \kappa) \approx \Delta_0(i, x) + \Delta_1(i, x)'(\kappa - \hat{\kappa}) + \frac{1}{2}(\kappa - \hat{\kappa})' \Delta_2(i, x)(\kappa - \hat{\kappa})$$

where $\Delta_0(i, x)$ is a scalar function of i and x , $\Delta_1(i, x)$ is a 2×1 vector of first derivatives, and $\Delta_2(i, x)$ is a 2×2 matrix of second derivatives. For any given i and x , these ‘ Δ ’ terms are parameters that are easily computed by Monte Carlo

⁴ For example, with $m=10000$ simulation paths in each stage for each of the mortality state variables, this would require 200 million simulation paths for the mortality state variables alone; combined with all the other calculations required, this implies a computational burden that is not for practical purposes feasible under real-time constraints.

⁵ The purpose of the probit transformation is to increase the domain of the function from $[0, 1]$ to the full real line and hence eliminate potential boundary problems.

simulation.⁶ Once we have these, simulated time- i expected survivor rates out to $i=50$ years can be recovered from

$$(8) \quad E[S(T+i, x) / S(T, x) | \kappa_1^j(T), \kappa_2^j(T)] \approx \\ \Phi\left(\Delta_0(i, x) + \Delta_1(i, x)'(\kappa^j - \hat{\kappa}) + \frac{1}{2}(\kappa^j - \hat{\kappa})' \Delta_2(i, x)(\kappa^j - \hat{\kappa})\right)$$

Each simulated $E[S(T+i, x) / S(T, x) | \kappa_1^j(T), \kappa_2^j(T)]$ can then be plugged into (6) to give us the corresponding simulated future annuity price we are seeking.

V. THE DISTRIBUTION OF FUTURE ANNUITY PRICES UNDER INTEREST-RATE AND LONGEVITY RISKS: SOME ILLUSTRATIVE RESULTS

We now provide an example based on a deferred annuity with a starting age of 65 and purchased at the age of 25, a current instantaneous spot interest rate equal to 4% and 10000 simulation trials. The annuity makes level payments of \$1 for each year the annuitant survives, and the annuity price is assumed to have a loading factor of 10%. Results are also presented for two versions of the mortality model: a version that assumes that the parameters of the mortality model are estimated with certainty (the

⁶ For more on the implementation of this approach see Cairns (2007), who also presents results suggesting that the approximations give fairly accurate results.

PC case), and a version that takes account of uncertainty in those parameters (the PU case).^{7,8}

We now examine three different cases. In the first case, we allow for longevity risk but not interest-rate risk: we model future longevity improvements using simulations from our mortality model, but we take the future instantaneous spot interest rate to be equal to its current value of 4%. In the second case, we assume that there are no changes in future longevity but we allow the instantaneous spot interest rate to be stochastic and use our interest-rate model to simulate its value at $T=40$. In the third case, we allow both interest rates and longevity to evolve stochastically over the period to $T=40$. In all cases, the future values of annuity prices are obtained by taking the $T=40$ present value of later cashflows discounted at the relevant spot interest rate, where these latter rates are obtained from the calibrated CIR interest rate model. Our results are presented in Figures 2-4 and Table 2.

INSERT FIGURE 2 HERE

INSERT FIGURE 3 HERE

INSERT FIGURE 4 HERE

INSERT TABLE 2 HERE

⁷ The parameter certain case uses the estimated values of the μ and C parameters, whereas the parameter uncertain case makes use of simulated values of these parameters drawn from the appropriate distributions. More details of the model and the simulation procedures (including the method used to allow for parameter uncertainty) are given in Cairns *et alia* (2006).

⁸ The simulations reported in this paper did not allow for any difference between the real-world and risk-neutral (or pricing) probability measures. We did not allow for this difference for two reasons. First, we do not have hard empirical evidence on what the relevant market prices of risk in the mortality model would be, and we need this to specify the risk-neutral probability measure; and, second, some illustrative results presented in Cairns *et alia* (2006) suggest that the prices of the longevity bond examined in their paper are very insensitive to the market prices of risk assumed there. However, the pricing calculations can easily be adjusted to allow for risk-neutral pricing if we specify what the risk-neutral probability measure should be.

Figure 2 shows the histogram of simulated future annuity prices if we allow for longevity risk but not interest-rate risk, Figure 3 shows the histogram of simulated future annuity prices if we allow for interest-rate risk but not longevity risk, and Figure 5 shows the histogram of simulated future annuity prices if we allow for both these risks simultaneously.

Table 2 shows that the current fair value of an annuity for a 65-year old male is 13.050 if we take the parameters of the mortality model to be certain, and 13.141 if we allow for uncertainty in the estimates of those parameters. These values provide benchmarks against which we can assess the prospective annuity prices that our current 25-year old might face when he reaches 65.

The first two columns in the Table give the main features of the distribution of future annuity prices 40 years' hence in the presence of longevity risk but no interest rate risk. They show that future annuity prices have a mean of 16.208 if we take the mortality parameters as certain (the PC case), and a mean of 16.044 if we allow for them to be uncertain (the PU case). These are respectively 24.2% and 22.1% higher than the prices of comparable annuities for 65-year olds bought now. Future annuity prices are expected to rise because the model projects further longevity improvements in the future. Amongst other results, the same column also shows that future annuity prices have an 80% confidence interval equal to [15.356 16.958] for the PC case and a somewhat wider 80% confidence interval of [14.449 17.422] in the PU case. (The corresponding standard deviations are 0.666 and 1.253.) These results indicate that longevity risk has a considerable impact on both the mean future annuity price and the

dispersion of future annuity prices, and that the degree of dispersion increases as we take account of parameter uncertainty.

Columns 3 and 4 give the comparable results for the opposite case where we allow for interest-rate risk but not longevity risk. The mean future annuity prices are now 13.135 (in the PC case) and 13.227 (in the PU case), which are much closer to the current annuity prices. Thus, allowing for interest-rate risk on its own has a much smaller impact on expected future annuity prices – in fact, it leads the expected price to rise by only 0.7% in each case – than allowing for longevity risk on its own. This should perhaps not be surprising given the mean-reversion embodied in the CIR model (4). The 80% confidence intervals are now [11.428 14.474] and [11.504 14.579] for the PC and PU cases respectively. (The corresponding standard deviations are about 1.2 in both cases.) So although interest-rate risk on its own has a negligible effect on the mean future annuity price, it has an impact on the dispersion of future annuity prices broadly comparable to that of longevity risk.

Finally, columns 5 and 6 show the results when we allow for both longevity and interest-rate risk. The mean future annuity prices now rise to 16.321 and 16.155: the impact of both longevity risk and interest-rate risk is to increase expected future annuity prices by 25.1% in the PC case and 22.9% in the PU case, relative to current prices. The 80% confidence intervals are now [13.952 18.323] and [13.362 18.580] for the PC and PU cases, and the corresponding standard deviations are 1.744 and 2.037.

So the combined effect of longevity and interest-rate risks is to considerably widen the dispersion of future annuity prices, in comparison with the cases in which each is treated separately. The mean future annuity price is also considerably higher

than the current annuity price, but this is principally due to projected future improvements in longevity rather than any effects of interest rate risk.⁹

VI. CONCLUSIONS AND IMPLICATIONS FOR DC PENSION PLAN MEMBERS

This paper proposes a simple procedure for estimating the distribution of future annuity prices in the presence of both longevity risk and interest-rate risk. Some illustrative results suggest that annuity prices are set to increase considerably, and that future annuity prices are also highly uncertain.

It is helpful if we end by elaborating a little on the implications of these findings for those in the early stages of a DC/401k pension plan, such as our illustrative 25-year old male plan member. Let us suppose that the value of the accumulated pension fund is given by F . For an individual aged 65 and retiring now, Table 2 tells us that his annual retirement income would be $F/13.141$ if we use the uncertain-parameter annuity valuation. However, our current 25-year old is likely to be much less fortunate in terms of his retirement income. Since he faces an expected future annuity price of 16.155 (if we use with the parameter-uncertain valuation and allow for both longevity and interest-rate risk), his expected retirement income is only $F/16.155$ – or 18.7% lower, other things being equal. This reduction in expected

⁹ The skewness results in Table 2 also show that the distribution of future annuity prices has a fairly strong negative skew, and the degree of skewness is greatest when we consider both longevity risk and interest-rate risk simultaneously.

retirement income is due primarily to projected longevity improvements over the course of his working lifetime.¹⁰

But worse still, his pension also becomes more risky. This riskiness can be seen in large degree of dispersion in the distribution of future annuity prices, and in the fact that this distribution has a positive skew and a very long right-hand tail. If we examine the quantiles of this tail, there is a 20% probability of a price in excess of 17.903, a 10% probability of a price in excess of 18.580, and so forth. Translated into their retirement-income equivalents, there is a 20% probability that our 25-year old will have a retirement income that is at least 26.6% (i.e., $13.141/17.903 - 1$) lower than that received by a male retiring now, and there is a 10% probability that he will receive an income that is at 29.3% (i.e., $13.141/18.580 - 1$) lower than his older counterpart, other things remaining equal. These amount to a gloomy prognosis for a risk-averse pension plan member.¹¹ Still, the good news (such as it is) is that although the news is always bad, it is not always quite so bad: if we look at the other, more fortunate, tail of the distribution, there is also a 10% probability that he will get a

¹⁰ Perhaps the most obvious response to this reduction would be for him to anticipate working longer: after all, if he is anticipating living longer, it might be reasonable for him to be prepared to work longer. The alternative to this, of course, is contribute around nearly 20% more to his pension plan than his 40-year older compatriot did.

¹¹ Our results about the riskiness of DC/401k pensions are however open to two offsetting sources of bias. On the one hand, as mentioned in note 2, the distribution of future instantaneous spot interest rates is likely to be under-estimated by the choice of a CIR interest-rate process, and a more empirically plausible interest rate process would lead the distribution of future annuity prices to become even more dispersed. On the other hand, we have implicitly assumed that the pension fund value, F , is fixed. In practice, it is much more likely that F would be stochastic and positively correlated with long-term interest rates. In this case, higher interest rates would be likely to produce both a high value of F and higher annuity prices, and DC pension outcomes would be more stable and less dispersed than we have suggested. We thank Tony Webb for this latter point.

retirement income that is almost as good as that received by the older pensioner now. He might just get lucky.¹²

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¹² A natural extension to our work is to look at the impact of inflation risk on real annuity prices and real DC pension outcomes, and we are currently on an extension of the present analysis to allow for inflation-rate risk.

TABLES

Table 1: Properties of the Probability Density Function of the Cox-Ingersoll-Ross Instantaneous Spot Interest Rate in 40 Years' Time

Parameter	Parameter Value
Mean	0.040
Standard deviation	0.032
Skewness	1.581
Kurtosis	6.742
10 th percentile	0.008
20 th percentile	0.014
30 th percentile	0.020
40 th percentile	0.026
50 th percentile	0.032
60 th percentile	0.040
70 th percentile	0.049
80 th percentile	0.061
90 th percentile	0.082

Note. The instantaneous spot interest rate is assumed to be governed by a CIR process (given by equations (4)-(5) in text) with parameters $a = 0.20$, $c = 0.10$ and $\bar{r} = 0.04$.

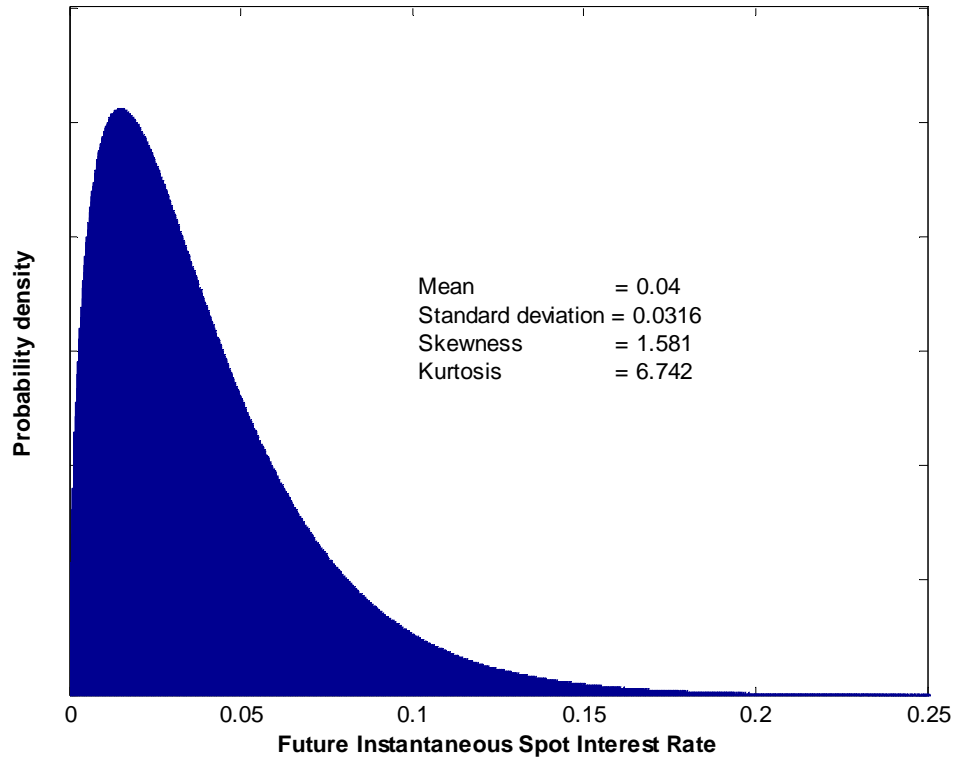
Table 2: Current Annuity Prices and the Probability Density Function of Annuity Prices in 40 Years' Time¹

Inputs						
	Age at retirement					65
	Years to retirement					40
	Current year					2007
	Current instantaneous spot interest rate					0.04
	Loading factor in annuity price					0.10
	Number of simulation trials					10000
Results for current annuity price						
	Current annuity price (PC ²)					13.050
	Current annuity price (PU ³)					13.141
Results for annuity prices at T=40						
Parameters of annuity price distribution	With longevity risk ⁴ but no interest rate risk ⁵		With interest rate risk ⁶ but no longevity risk		With longevity risk ³ and interest rate risk ⁶	
	PC	PU	PC	PU	PC	PU
Mean	16.208	16.044	13.135	13.227	16.321	16.155
Std	0.666	1.253	1.234	1.245	1.744	2.037
Skewness	-0.952	-1.108	-1.121	-1.121	-0.8278	-0.571
Kurtosis	4.484	5.883	4.391	4.389	3.746	3.294
10 th perc	15.356	14.449	11.428	11.504	13.952	13.362
20 th perc	15.735	15.171	12.203	12.286	14.938	14.482
30 th perc	15.968	15.620	12.693	12.781	15.654	15.243
40 th perc	16.146	15.940	13.082	13.174	16.158	15.871
50 th perc	16.298	16.227	13.398	13.493	16.591	16.384
60 th perc	16.431	16.476	13.685	13.782	16.990	16.879
70 th perc	16.581	16.731	13.949	14.049	17.392	17.367
80 th perc	16.739	17.018	14.208	14.310	17.824	17.903
90 th perc	16.957	17.422	14.474	14.579	18.323	18.580

Notes: 1. In all cases discounting is carried out at the spot interest rate for the relevant maturity as determined by the calibrated CIR interest-rate model. 2. 'PC' means that the simulations assume the parameters of the stochastic mortality model to be certain. 3. 'PU' means that the simulations allow for uncertainty in the parameters of the stochastic mortality model using the method outlined in Cairns *et alia* (2006). 4. Longevity risk is modelled using simulated values of ${}_1(40)$ and ${}_2(40)$ obtained using the Cairns, Blake and Dowd (CBD, 2006) stochastic mortality model and taking account of interim stochastic mortality improvement in the period since 2002. These parameter values are based on estimates of the mortality of English and Welsh males aged 65 over the period 1982-2002. See equations (1)-(2) in text. 5. The instantaneous spot interest rate at $T=40$ is assumed to be equal to 0.04. 6. The instantaneous spot interest rate and term structure of interest rates are assumed to be governed by a CIR process (given by equation (6) in text) with parameters $\alpha = 0.20$, $c = 0.10$ and $\bar{r} = 0.04$.

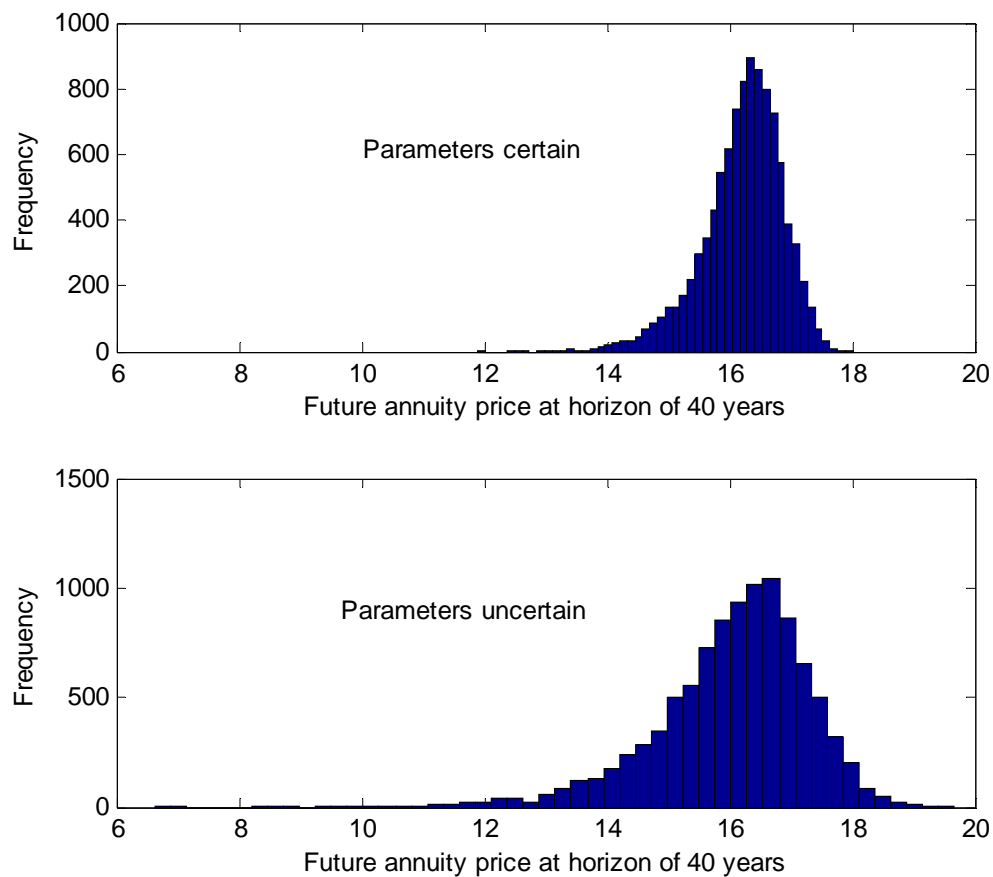
FIGURES

Figure 1: Density Function for Future CIR Instantaneous Spot Interest Rate at a 40-Year Horizon



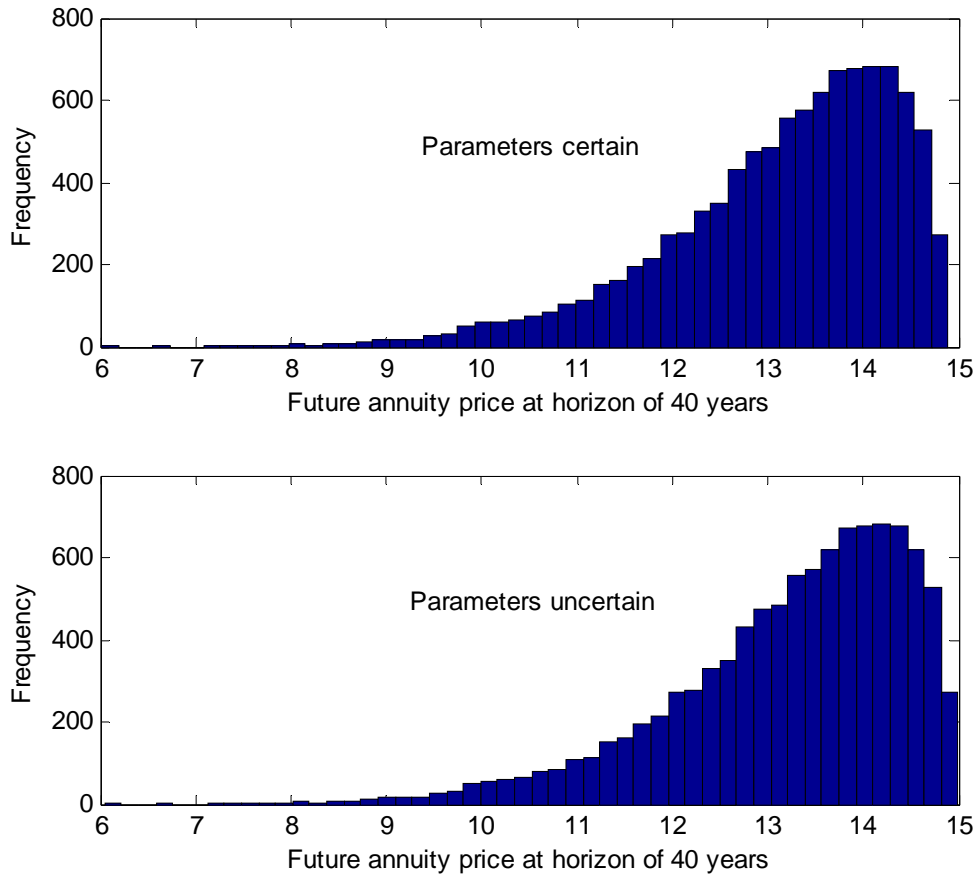
Notes: Figures shows a non-central chi-squared distribution based on (4) calibrated to $\alpha = 0.20$, $c = 0.10$, $\bar{r} = 0.04$, $r(0) = 0.04$ and $T = 40$.

Figure 2: Histogram of Simulated Future Annuity Prices under Longevity Risk but no Interest-Rate Risk



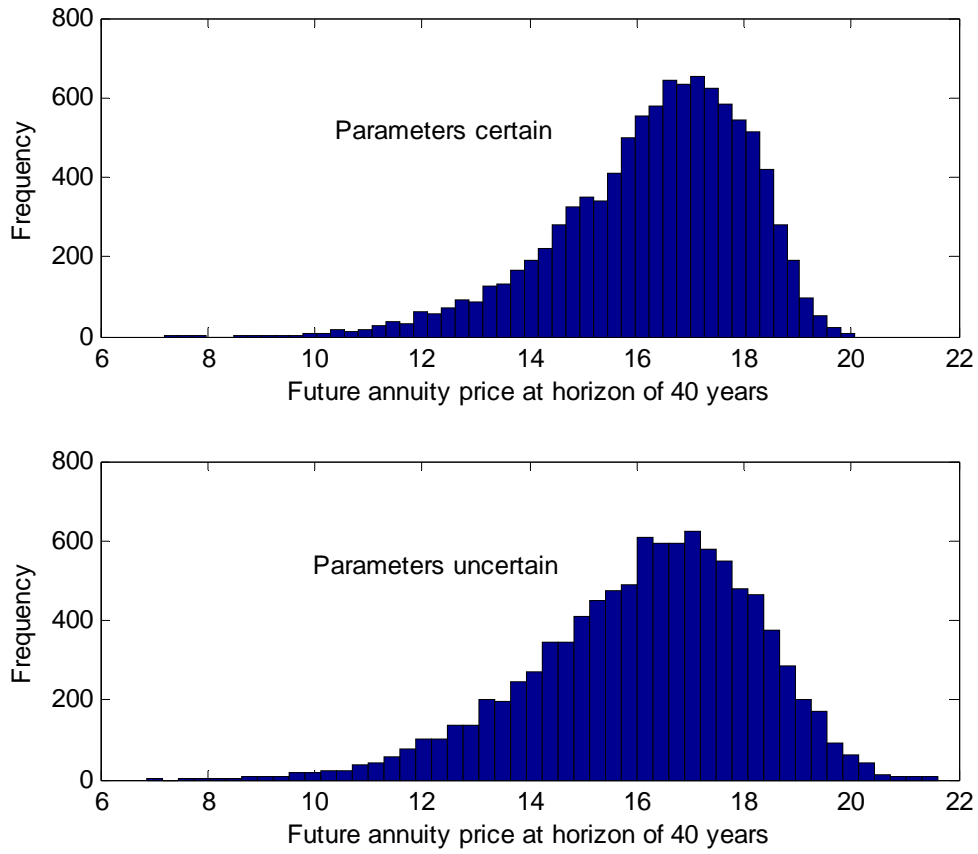
Notes: As per Notes 2, 3, 4 and 5 to Table 2.

Figure 3: Histogram of Simulated Future Annuity Prices under Interest-Rate Risk but no Longevity Risk



Notes: As per Notes 2, 3, and 6 to Table 2.

Figure 4: Histogram of Simulated Future Annuity Prices under Longevity Risk and Interest-Rate Risk



Notes: As per Notes 2, 3, and 4 to Table 2.