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Estimating the Distribution of Future Annuity  
Values under Interest-Rate and Longevity  
Risks

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Annuity Values under Interest-Rate and Longevity Risks**

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# **A Computationally Efficient Algorithm for Estimating the Distribution of Future Annuity Values under Interest-Rate and Longevity Risks**

## **Abstract**

This paper proposes a computationally efficient algorithm for quantifying the impact of interest-rate risk and longevity risk on the distribution of annuity values in the distant future. The algorithm simulates the state variables out to the end of the horizon period and then uses a Taylor series approximation to compute approximate annuity values at the end of that period, thereby avoiding a computationally expensive ‘simulation-within-simulation’ problem. Illustrative results suggest that annuity values are likely to rise considerably, but are also quite uncertain. These findings have some unpleasant implications for both defined contribution pension plans and for defined benefit plan sponsors considering using annuities to hedge their exposure to these risks at some point in the future.

JEL Classification Numbers: G23, C15, C53

Key words: longevity risk, interest-rate risk, annuity values, Taylor series approximation, pension risk

## I. INTRODUCTION

This paper examines the impact of interest-rate risk and longevity risk on the distribution of annuity values in the distant future. At first sight, this might appear to be a rather arcane issue. Most people do not care much about current annuity values, so why would they care about possible annuity values in, say, 40 years' time? None the less, we would suggest that the future values of annuities are much more important than they might first appear to be. One reason is suggested by the global increase in longevity: people are living longer than previously anticipated, and it is natural to ask how much this is likely to cost. To answer this question, we need some index of the cost of life expectancy, and a natural index is the cost of a life annuity, i.e., the expected present value of an income stream of \$1 per annum until the buyer of the annuity dies. The cost of increased longevity is then the likely increase in future annuity values. Thus, an assessment of likely future annuity values is a key to evaluating the likely cost of higher longevity.

There is a second and often more personal reason why we should care about future annuity values. Consider the illustrative case of a male currently aged 25 who is starting a defined contribution (DC) pension plan and is planning to retire in, say, 40 years' time at the age of 65. He anticipates that when he reaches that age, he will convert his accumulated pension fund into a life annuity in order to hedge his own longevity risk and avoid outliving his own financial resources. The value of his retirement income will depend not only on the value of his pension fund, but also on the price of annuities at that time. Other things being equal, this means that his retirement income prospects will be affected by the distribution of future annuity

values: the greater the dispersion of that distribution, the more risky his retirement income will be. Hitherto, analyses of DC plans have tended to focus on the risks facing the member that arise from the uncertainty of the future value of the pension fund itself, i.e., they have tended to focus on investment and contribution risks during the accumulation stage of the plan, and, in so doing, have tended to overlook the impact of distribution stage risks, most notably interest rate and longevity risks, that can affect retirement income through their impact on annuity values. The superficially arcane issue of the distribution of future annuity values therefore turns out to be a key ingredient in determining the risks associated with DC pension plans. The same holds for defined benefit plan sponsors who consider using annuities to hedge their exposure to interest-rate and longevity risks at some point in the future.

This paper investigates this issue further and provides some illustrative results for a long-term (40-year) horizon. As is well-known, the principal factors that determine annuity values are life-expectancy prospects (which depend on future mortality rates) and interest rates.<sup>1</sup> To model future annuity values, we therefore need a model of future mortality rates. We also need a model of future interest rates, since these will influence the future price of the bonds the annuity provider buys in order to make annuity payments and, in turn, the discount rate used to value the annuity itself. The models we use involve stochastic simulation, but any naïve attempt to use stochastic simulation runs into a problem: if we simulate the state variables out to some future horizon period  $T$ , we then face the problem of how to obtain the future time- $T$  annuity values, contingent on the time- $T$  values of the state variables. The most

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<sup>1</sup> If interest rates are low at the point of retirement, the price of the annuity purchased will be high – equivalently the yield on the annuity will be low – since the annuity provider buys bonds to make the annuity payments and these will be expensive when interest rates are low.

obvious approach is to use stochastic simulation for this purpose, but we would then find ourselves running ‘simulations within simulations’ which can be computationally very expensive. To operate within feasible real-time constraints, we need to find some alternative method to obtain the future annuity values contingent on the outcomes of the time- $T$  state variables. We solve this problem using a Taylor series approximation: we simulate the state variables out to  $T$ , and then use the Taylor series approximation to estimate the annuity values at  $T$  as functions of the values of the state variables.

This paper is organized as follows. Section 2 describes the stochastic mortality model used to model longevity risk, and section 3 outlines the stochastic interest-rate model that we use.<sup>2</sup> Section 4 explains computational issues and proposes a ‘Taylor series approximation within simulation’ approach that allows us to estimate future annuity values using simulated future values of the underlying state variables. Section 5 presents some illustrative results, and Section 6 concludes and elaborates on the implications of our findings for the riskiness of DC plan retirement incomes.

## II. A STOCHASTIC MORTALITY MODEL

We model mortality stochastically using the Cairns-Blake-Dowd (CBD) model described in Cairns *et alia* (2006a) but with the parameterisation defined in Cairns *et alia* (2009). Let  $q(t+1, x)$  be the realized mortality rate in year  $t+1$  (that is, from

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<sup>2</sup> The present paper assumes that longevity and interest-rate risks are independent of each other. We believe that this is a reasonable assumption for most normal states of the world in the short run. We recognize that there will be some extreme states – such as those associated with a global war, a pandemic or a huge meteorite hitting Earth – where the assumption is not valid. In the long run, however, increasing longevity, if it is associated with an increase in the length of the working life, will increase the supply of labour relative to capital which, in turn, will increase the return on capital (i.e., the rate of interest rate) relative to the wage rate.

time  $t$  to time  $t+1$ ) for individuals aged  $x$  at time  $t$ . We assume that  $q(t+1, x)$  is governed by a two-factor Perks stochastic process:

$$(1) \quad q(t+1, x) = \exp[\kappa_1(t+1) + \kappa_2(t+1)(x - \bar{x})] / \{1 + \exp[\kappa_1(t+1) + \kappa_2(t+1)(x - \bar{x})]\}$$

where  $\kappa_1(t+1)$  and  $\kappa_2(t+1)$  are themselves stochastic processes that are measurable at time  $t+1$  (see Perks, 1932, Benjamin and Pollard, 1993), and  $\bar{x}$  is a constant that is typically set to the mean of the range of ages used to calibrate the model. Now let  $\kappa(t+1) = (\kappa_1(t+1), \kappa_2(t+1))'$  and assume that  $\kappa(t+1)$  is a random walk with drift:

$$(2) \quad \kappa(t+1) = \kappa(t) + \mu + CZ(t+1)$$

where  $\mu$  is a constant  $2 \times 1$  vector of drift parameters,  $C$  is a constant  $2 \times 2$  lower triangular Choleski square root matrix of the covariance matrix  $V$  (that is,  $V = CC^T$ ), and  $Z(t+1)$  is a  $2 \times 1$  vector of independent standard normal variables. Cairns *et alia* (2006a, 2009) show that this model provides a good fit to UK Office for National Statistics (ONS) data for English and Welsh males over 1961-2004.

Now let  $S(t+1, x)$  be the survivor index at time  $t+1$  of a cohort aged  $x$  in year 0: that is,  $S(t+1, x)$  is the probability, measured retrospectively, that an individual aged  $x$  at time 0 survives to time  $t$ . For any given  $x$ ,  $S(0, x) = 1$  and  $S(t+1, x)$  will decrease as  $t$  gets bigger and eventually approach 0 as  $t$  gets large. Given any path of  $q(t+1, x)$  as obtained above, we then obtain a corresponding

path of  $S(t+1, x)$  from the relationship between mortality rates and the survivor index:

$$(3) \quad S(t+1, x) = (1 - q(t+1, x))S(t, x)$$

Note that these survivor rates are driven off the state variables  $\kappa_1(t+1)$  and  $\kappa_2(t+1)$ . For our purposes, we wish to simulate sets of state variables out to a future time  $T$ , and then estimate the expectations of (3) conditional on surviving to the specified future date and conditional on the future values of the state variables  $\kappa_1(T)$  and  $\kappa_2(T)$  at that date.

### III. A STOCHASTIC INTEREST-RATE MODEL

We also need a model of the interest-rate process, and the simplest model that meets our requirements is the Cox-Ingersoll-Ross (CIR) model (1985). This model postulates that the instantaneous spot interest rate  $r$  obeys the following continuous-time process:

$$(4) \quad dr(t) = \alpha(\bar{r} - r(t))dt + \sigma\sqrt{r(t)}dW(t)$$

where  $\alpha$  indicates the strength of the mean-reversion process governing  $r$ ,  $\bar{r}$  is the long-term mean instantaneous spot interest rate,  $\sigma$  is the interest-rate volatility and  $dW(t)$  is a standard geometric Brownian motion. This model is attractive because it

allows for interest rates to be mean-reverting but does not allow them to become negative.<sup>3</sup> Another attractive feature is that it gives us a straightforward formula for the spot rate term structure based on the current instantaneous spot rate: if  $R(t, T)$  is the time- $t$  spot rate for the fixed maturity date  $T$  (that is,  $T-t$  years to maturity), then

$$(5) \quad R(t, T) = -(T-t)^{-1} \ln P(t, T)$$

where  $P(t, T)$  is the time- $t$  price of a zero-coupon bond with maturity  $T$  and where

$$P(t, T) = A(t, T) \exp[-B(t, T)r(T)]$$

$$A(t, T) = \left( 2\gamma \exp[(\alpha + \gamma)(T-t)/2] / \{(\gamma + \alpha)(\exp[\gamma(T-t)] - 1) + 2\gamma\} \right)^{2\bar{r}\alpha/\sigma^2}$$

$$B(t, T) = 2(\exp[\gamma(T-t)] - 1) / \{(\gamma + \alpha)(\exp[\gamma(T-t)] - 1) + 2\gamma\}$$

$$\gamma = \sqrt{\alpha^2 + 2\sigma^2}$$

From a computational perspective, the CIR model is appealing because the exact distribution of the instantaneous spot interest rate under the CIR model is known. To be precise, if  $r(T)$  follows a CIR process, then  $(4\alpha r(T)) / \{\sigma^2(\exp[\alpha T] - 1)\}$  has a non-central chi-squared distribution with  $4\alpha\bar{r} / \sigma^2$  degrees of freedom and a non-centrality parameter equal to  $(4\alpha r(0)) / \{\sigma^2(\exp[\alpha T] - 1)\}$  (Cairns, 2004, Theorem 4.8 (c)). This means that we can simulate values of  $r(T)$  directly from their exact distribution using the CIR parameters and the current instantaneous spot rate  $r(0)$  as

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<sup>3</sup> The model and its properties are discussed in Cairns (2004, p.67).

inputs. Once we have these terminal instantaneous spot rate values, we can then use (4) to infer the spot-rate term structures contingent on each of these values.

A typical distribution for  $r(T)$  is given in Figure 1 and Table 1, based on assumed values of  $\alpha = 0.20$ ,  $\sigma = 0.10$  and  $\bar{r} = 0.04$ . Figure 1 shows the pdf and Table 1 gives some of its key parameters. We can see that the pdf has a strong positive skew and a long right-hand tail. What is perhaps most striking about this pdf is the extent of its dispersion – for example, 30% of simulated values are under 0.02, 30% are above 0.049 and we get a small number of very high values in the long right-hand tail – and this is the case even though the spot-rate process is mean-reverting.<sup>4</sup>

INSERT FIGURE 1

INSERT TABLE 1

These findings confirm that interest-rate risk will have a considerable impact on the distribution of future annuity values.

#### IV. COMPUTATIONAL ISSUES

If we combine these models, we have three random state variables: the  $\kappa_1(\cdot)$  and  $\kappa_2(\cdot)$  state variables from the mortality model, and the instantaneous spot interest rate from  $r(\cdot)$  from the interest-rate model. Suppose that the current time is time 0

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<sup>4</sup> It is also worth noting that the CIR process is widely regarded by interest-rate practitioners as *under-*estimating the true distribution of instantaneous spot rates at long horizons. To the extent that this is the case, then our later results will under-state the impact of interest-rate risk on the distribution of future annuity values and therefore under-estimate the riskiness of DC pension plans.

and we wish to simulate these state variables out to some future period  $T$ . Suppose, then, that we take  $J$  simulation paths of each state variable out to period  $T$ , and let  $[\kappa_1^j(T), \kappa_2^j(T), r^j(T)]$  be the  $j^{\text{th}}$  set of simulated state variables for period  $T$ . The fair value of an annuity at time  $T$ ,  $a(T)$ , then depends on the values of these state variables, so we will, from time to time, use the extended notation  $a(T) \equiv a(T; \kappa_1(T), \kappa_2(T), r(T))$  to reflect the fair value's dependence on the state variables. Similarly, in the case of interest rates, we have  $R(T, T+i) \equiv R(T, T+i; r(T))$ . The fair value of the annuity, conditional on the simulated state variables under simulation path  $j$  is thus

$$(6) \quad a^j(T) = a\left(T; \kappa_1^j(T), \kappa_2^j(T), r^j(T)\right) = \\ (1 + \phi) \sum_{i=1}^{50} \exp\left(-iR(T, T+i; r^j(T))\right) E[S(T+i, x) / S(T, x) | \kappa_1^j(T), \kappa_2^j(T)]$$

where  $\phi$  is the cost loading factor built into the annuity value and  $R(T, T+i; r(T))$  is the spot interest rate prevailing over the period from  $T$  to  $T+i$  given  $r(T)$ . Equation (6) is a sum of the product of future expected survivor rates from age 66 to age 115 (conditional on surviving to age 65) and the price of a zero-coupon bond maturing with unit value at the same age, and we assume that no-one lives beyond age 115. The term  $E[S(T+i, x) / S(T, x) | \kappa_1^j(T), \kappa_2^j(T)]$  is to be interpreted as the expected probability that an individual currently aged  $x$  will survive to year  $T+i$  conditional on their surviving to  $T$  and conditional on the mortality state parameters at  $T$ .

However, we cannot compute (6) directly because there is no simple formula for the  $E[S(T+i, x)/S(T, x) | \kappa_1(T), \kappa_2(T)]$  in terms of the mortality state variables  $\kappa_1^j(T)$  and  $\kappa_2^j(T)$ . The most obvious solution to this problem would be to use stochastic simulation to estimate  $E[S(T+i, x)/S(T, x) | \kappa_1(T), \kappa_2(T)]$ , for each set of  $\kappa_1^j(T)$  and  $\kappa_2^j(T)$ . For example, we might use  $M$  simulation trial paths to generate  $j=1, \dots, M$  pairs of simulated time- $T$  state variables  $\kappa_1^j(T)$  and  $\kappa_2^j(T)$ . From each such pair, we might then generate  $k=1, \dots, M$  sets of paths for the post- $T$  survivor rates  $S^{jk}(T+i, x)/S^{jk}(T, x)$  and obtain  $E[S^j(T+i, x)/S^j(T, x) | \kappa_1^j(T), \kappa_2^j(T)]$  as their average. In principle, this solution would work, but it involves us taking  $M^2$  simulation paths and this would be computationally extremely expensive.<sup>5</sup>

A simple way to reduce this computational burden is to use a Taylor series approximation, as suggested by Cairns (2007). Let  $\kappa = [\kappa_1(T), \kappa_2(T)]'$  and let  $\hat{\kappa} = E[\kappa]$  be the expectation of the mortality state variables at  $T$ . Define  $f(i, x, \kappa) = \Phi^{-1}(E[S(T+i, x)/S(T, x) | \kappa = (\kappa_1(T), \kappa_2(T))])$  as the probit transformation of  $E[S(T+i, x)/S(T, x) | \kappa = (\kappa_1(T), \kappa_2(T))]$ , where  $\Phi(\cdot)$  is the standard normal distribution function.<sup>6</sup> We then take the following second-order Taylor series expansion of  $f(i, x, \kappa)$  around  $\hat{\kappa}$ :

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<sup>5</sup> For example, with  $M=10000$  simulation paths in each stage for each of the mortality state variables, this would require 200 million simulation paths for the mortality state variables alone; combined with all the other calculations required, this implies a computational burden that is not for practical purposes feasible under real-time constraints.

<sup>6</sup> The purpose of the probit transformation is purely to increase the domain of the function from  $[0, 1]$  to the full real line and hence eliminate potential boundary problems.

$$(7) \quad f(i, x, \kappa) \approx \Delta_0(i, x) + \Delta_1(i, x)'(\kappa - \hat{\kappa}) + \frac{1}{2}(\kappa - \hat{\kappa})' \Delta_2(i, x)(\kappa - \hat{\kappa})$$

where  $\Delta_0(i, x)$  is a scalar function of  $i$  and  $x$ ,  $\Delta_1(i, x)$  is a  $2 \times 1$  vector of first derivatives, and  $\Delta_2(i, x)$  is a  $2 \times 2$  matrix of second derivatives. For any given  $i$  and  $x$ , these ‘ $\Delta$ ’ terms are parameters that are easily computed by Monte Carlo simulation.<sup>7</sup> Once we have these, simulated time- $i$  expected survivor rates out to  $i=50$  years can be recovered from

$$(8) \quad E[S(T+i, x) / S(T, x) | \kappa_1^j(T), \kappa_2^j(T)] \approx \Phi\left(\Delta_0(i, x) + \Delta_1(i, x)'(\kappa^j - \hat{\kappa}) + \frac{1}{2}(\kappa^j - \hat{\kappa})' \Delta_2(i, x)(\kappa^j - \hat{\kappa})\right)$$

Each simulated  $E[S(T+i, x) / S(T, x) | \kappa_1^j(T), \kappa_2^j(T)]$  can then be plugged into (6) to give us the corresponding simulated future annuity value we are seeking.

## V. THE DISTRIBUTION OF FUTURE ANNUITY VALUES UNDER INTEREST-RATE AND LONGEVITY RISKS: SOME ILLUSTRATIVE RESULTS

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<sup>7</sup> For more on the implementation of this approach. see Cairns (2007), who carries out numerical tests on the accuracy of the approximation.. For spot survival probabilities, the quadratic approximation delivers values that are typically within 0.2% (at a horizon of  $i=10$  years) or 1% ( $i=30$  years) in relative terms. Thus, the quadratic approximation is less good for longer maturities, but, nevertheless, still very accurate. Approximate values can be either above or below true values. However, a very small bias can creep in if  $\hat{\kappa}$  is not calibrated to the relevant time horizon, i.e.,  $\hat{\kappa} = E[\kappa(T)]$ . For annuity prices with a fixed rate of interest of 4%, the quadratic approximation delivers values that are within 0.2% of the true price. The percentage relative errors noted above assume a simulation time horizon of 40 years but with  $\hat{\kappa}$  calibrated to the 20-year expected values,  $\hat{\kappa} = E[\kappa(20)]$ , in order to test the robustness of the approximation relative to the particular choice of  $\hat{\kappa}$ . If we focused on a single simulation time horizon,  $T$ , then the errors above can be reduced by calibrating  $\hat{\kappa} = E[\kappa(T)]$  to that specific time horizon.

We now provide an example based on a deferred annuity with a starting age of 65 and purchased at the age of 25, a current instantaneous spot interest rate equal to 4% and 10000 simulation trials. The annuity makes level payments of \$1 for each year the annuitant survives, and the annuity value is assumed to incorporate a loading factor of 10%.<sup>8</sup> Results are also presented for two versions of the mortality model: a version that assumes that the parameters of the mortality model are estimated with certainty (the PC case), and a version that takes account of uncertainty in those parameters (the PU case).<sup>9, 10</sup>

We examine three different cases. In the first case, we allow for longevity risk but not interest-rate risk: we model future longevity improvements using simulations from our mortality model, but we take the future instantaneous spot interest rate to be equal to its current value of 4%. In the second case, we assume that there are no changes in future longevity, but we allow the instantaneous spot interest rate to be stochastic and use our interest-rate model to simulate its value at  $T=40$ . In the third case, we allow both interest rates and longevity to evolve stochastically over the period to  $T=40$ . In all cases, the future annuity values are obtained by taking the  $T=40$  present value of later cashflows discounted at the relevant spot interest rate, where

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<sup>8</sup> This is approximately equal to the average loading factor found by Finkelstein and Poterba (2002) for UK insurers selling level annuities.

<sup>9</sup> The parameter certain case uses the estimated values of the  $\mu$  and  $C$  parameters, whereas the parameter uncertain case makes use of simulated values of these parameters drawn from the appropriate distributions. More details of the model and the simulation procedures (including the method used to allow for parameter uncertainty) are given in Cairns *et alia* (2006a).

<sup>10</sup> The simulations reported in this paper did not allow for any difference between the real-world and risk-neutral probability and pricing measures. We did not allow for this difference for two reasons. First, we do not have hard empirical evidence on what the relevant market price of risk in the mortality model would be, and we need this to specify the risk-neutral probability measure; and, second, some illustrative results presented in Cairns *et alia* (2006a) suggest that the prices of the longevity bond examined in their paper are fairly insensitive to the specific market prices of risk they assumed.

these latter rates are obtained from the calibrated CIR interest rate model. Our results are presented in Table 2 and Figures 2-4.

INSERT TABLE 2 HERE

Table 2 shows that the current fair value of an annuity for a 65-year-old male is 13.050 if we take the parameters of the mortality model to be certain, and 13.141 if we allow for uncertainty in the estimates of those parameters. These values provide benchmarks against which we can assess the prospective annuity prices that our current 25-year old might face when he reaches 65.

The first two columns in the Table give the main features of the distribution of future annuity values 40 years' hence in the presence of longevity risk but no interest rate risk. They show that future annuity values have a mean of 16.208 if we take the mortality parameters as certain (the PC case), and a mean of 16.044 if we allow for them to be uncertain (the PU case). These are, respectively, 24.2% and 22.1% higher than the values of comparable annuities for 65-year olds bought now. Clearly, future annuity values are expected to rise because the model projects further longevity improvements in the future. But what is less clear is the impact on future annuity values of longevity risk, i.e., the uncertainty attached to future longevity projections. Columns 1 and 2 show that future annuity values have an 80% confidence interval equal to [15.356 16.958] for the PC case and a somewhat wider 80% confidence interval of [14.449 17.422] in the PU case.<sup>11</sup> (The corresponding standard deviations are 0.666 and 1.253.) These results indicate that longevity risk has a considerable

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<sup>11</sup> The 80% confidence interval comes from comparing the difference between the 10<sup>th</sup> and 90<sup>th</sup> percentiles.

impact on both the mean future annuity value and the dispersion of future annuity values, and that the degree of dispersion increases as we take account of parameter uncertainty.

Columns 3 and 4 give the comparable results for the case where we allow for interest-rate risk but not longevity risk. The mean future annuity values are now 13.135 (in the PC case) and 13.227 (in the PU case), which are much closer to the current annuity values. Thus, allowing for interest-rate risk on its own has a much smaller impact on expected future annuity values – in fact, it leads the expected value to rise by only 0.7% in each case – than allowing for longevity risk on its own. This follows because of the mean-reversion embodied in the CIR model (4). The 80% confidence intervals are now [11.428 14.474] and [11.504 14.579] for the PC and PU cases respectively. (The corresponding standard deviations are about 1.2 in both cases.) So although interest-rate risk on its own has a negligible effect on the mean future annuity value, it has an impact on the dispersion of future annuity values broadly comparable to that of longevity risk (in the PU case, although not in the PC case)

Finally, columns 5 and 6 show the results when we allow for both longevity and interest-rate risk. The mean future annuity values now rise to 16.321 and 16.155: the impact of both longevity risk and interest-rate risk is to increase expected future annuity values by 25.1% in the PC case and 22.9% in the PU case, relative to current values. The 80% confidence intervals are now [13.952 18.323] and [13.362 18.580] for the PC and PU cases, and the corresponding standard deviations are 1.744 and 2.037.

Figure 2 shows the histogram of simulated future annuity values if we allow for longevity risk but not interest-rate risk, Figure 3 shows the same histogram if we allow for interest-rate risk but not longevity risk, and Figure 4 shows the histogram if we allow for both these risks simultaneously. Figure 4 indicates that the distribution of future annuity values has a fairly strong negative skew when we allow for both risks. Figure 2 reveals that the distribution has a very small negative skew if only longevity risk is considered. Figure 2 confirms that the cause of the negative skew in Figure 4 is predominantly caused by the negative skew in the distribution of future annuity values due to interest rate risk and this is the direct counterpart of the strong positive skew in the distribution of interest rates shown in Figure 1.

INSERT FIGURE 2 HERE

INSERT FIGURE 3 HERE

INSERT FIGURE 4 HERE

## VI. CONCLUSIONS AND IMPLICATIONS FOR DC PENSION PLAN MEMBERS

This paper proposes a simple computationally efficient algorithm for estimating the distribution of future annuity values in the presence of both longevity and interest-rate risk. The algorithm is based on a second-order Taylor series expansion of the probit transformation of the expected values of a survivor index in future years, conditional on an individual surviving to the previous year and conditional on the state parameters

governing the stochastic mortality model. The algorithm allows us to avoid the computational problem of simulation-within-simulation and hence increase computational efficiency by a factor of 10000 in the case where we wish estimate the distribution of future annuity prices using 10000 simulations.

Some illustrative results suggest that the combined effect of longevity and interest-rate risks is to considerably widen the dispersion of future annuity values, in comparison with the cases in which each is treated separately. The mean future annuity value is also considerably higher than the current annuity value, but, as we have seen, this is principally due to projected future improvements in longevity rather than to any effects of interest rate risk. Interest rate risk is largely responsible for giving the distribution of future annuity values a strong negative skew.

It is helpful if we end by elaborating a little on the implications of these findings for those in the early stages of a DC pension plan, such as our illustrative 25-year old male plan member. Let us suppose that the value of the accumulated pension fund is given by  $F$ . For an individual aged 65 and retiring now, Table 2 tells us that his annual retirement income would be  $F/13.141$  if we use the uncertain-parameter annuity valuation. However, our current 25-year old is likely to be much less fortunate in terms of his retirement income. Since he faces an expected future annuity value of 16.155 (if we use with the parameter-uncertain valuation and allow for both longevity and interest-rate risk), his expected retirement income is only  $F/16.155$  – or 18.7% lower, other things being equal. This reduction in expected retirement income is due

primarily to projected longevity improvements over the course of his working lifetime.<sup>12</sup>

But worse still, his pension also becomes more risky. One cause of this increase in risk is the dispersion in the distribution of future annuity values. Of particular concern is the positive tail of this distribution. If we examine the quantiles of this tail, there is a 20% probability of a value in excess of 17.903, a 10% probability of a value in excess of 18.580, and so forth. Translated into their retirement-income equivalents, there is a 20% probability that our 25-year old will have a retirement income that is at least 26.6% (i.e.,  $13.141/17.903 - 1$ ) lower than that received by a male retiring now, and there is a 10% probability that he will receive an income that is at 29.3% (i.e.,  $13.141/18.580 - 1$ ) lower than his older counterpart, other things remaining equal, in particular the size of the retirement fund.<sup>13</sup> These amount to a gloomy prognosis for a risk-averse pension plan member. Still, the good news (such as it is) is that although the news is always bad, it is not always quite so bad: if we look at the other, more fortunate, tail of the distribution, there is also a 10% probability that he will get a retirement income that is almost as good as that received by the older pensioner now. He might just get lucky.

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<sup>12</sup> Perhaps the most obvious response to this reduction would be for him to anticipate working longer: after all, if he is anticipating living longer, it might be reasonable for him to be prepared to work longer. The alternative to this, of course, is contribute around nearly 20% more to his pension plan than his 40-year older compatriot did.

<sup>13</sup> Our results about the riskiness of DC pensions are however open to two offsetting sources of bias. On the one hand, as mentioned in note 4, the distribution of future instantaneous spot interest rates is likely to be under-estimated by the choice of a CIR interest-rate process, and a more empirically plausible interest rate process would lead the distribution of future annuity prices to become even more dispersed. On the other hand, we have implicitly assumed that the pension fund value,  $F$ , is fixed. In practice, it is much more likely that  $F$  would be stochastic and positively correlated with long-term interest rates, particularly if the assets in the pension fund were dominated by bonds. In this case, higher interest rates would be likely to produce both a high value of  $F$  and higher annuity prices, and DC pension outcomes would be more stable and less dispersed than we have suggested. We thank Tony Webb for this latter point.

Finally, our analysis has identified an important need for pension plans and their members to hedge the interest-rate and longevity risks in their future annuity purchases. They could do this using annuity futures or annuity futures options as suggested in Blake *et alia* (2006). Unfortunately, these contracts do not yet exist, either in capital markets or over-the-counter form. But we believe this is only a matter of time. The framework outlined in Cairns *et alia* (2006b) shows how these contracts can be priced, while the computationally efficient algorithm outlined above would allow the contract pricing to be done in real time.

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## TABLES

**Table 1: Properties of the Probability Density Function of the Cox-Ingersoll-Ross Instantaneous Spot Interest Rate in 40 Years' Time**

Parameter	Parameter Value
Mean	0.040
Standard deviation	0.032
Skewness	1.581
Kurtosis	6.742
10 <sup>th</sup> percentile	0.008
20 <sup>th</sup> percentile	0.014
30 <sup>th</sup> percentile	0.020
40 <sup>th</sup> percentile	0.026
50 <sup>th</sup> percentile	0.032
60 <sup>th</sup> percentile	0.040
70 <sup>th</sup> percentile	0.049
80 <sup>th</sup> percentile	0.061
90 <sup>th</sup> percentile	0.082

Note. The instantaneous spot interest rate is assumed to be governed by a CIR process (given by equations (4)-(5) in text) with parameters  $\alpha = 0.20$ ,  $\sigma = 0.10$  and  $\bar{r} = 0.04$ .

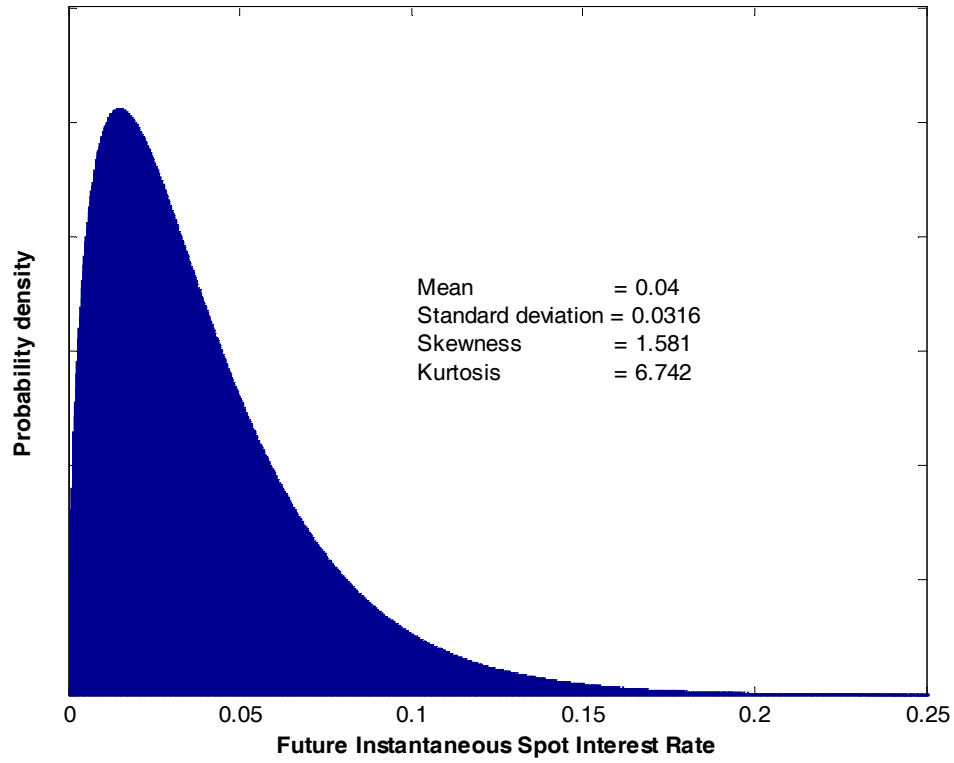
**Table 2: Current Annuity Values and the Probability Density Function of Annuity Values in 40 Years' Time<sup>1</sup>**

<b>Inputs</b>						
	Age at retirement					65
	Years to retirement					40
	Current year					2007
	Current instantaneous spot interest rate					0.04
	Loading factor in annuity value					0.10
	Number of simulation trials					10000
<b>Results for the current annuity value</b>						
	Current annuity value (PC <sup>2</sup> )					13.050
	Current annuity value (PU <sup>3</sup> )					13.141
<b>Results for annuity values at T=40</b>						
Parameters of annuity value distribution	With longevity risk <sup>4</sup> but no interest rate risk <sup>5</sup>		With interest rate risk <sup>6</sup> but no longevity risk <sup>7</sup>		With longevity risk <sup>4</sup> and interest rate risk <sup>6</sup>	
	PC	PU	PC	PU	PC	PU
Mean	16.208	16.044	13.135	13.227	16.321	16.155
Std	0.666	1.253	1.234	1.245	1.744	2.037
Skewness	-0.952	-1.108	-1.121	-1.121	-0.8278	-0.571
Kurtosis	4.484	5.883	4.391	4.389	3.746	3.294
10 <sup>th</sup> perc	15.356	14.449	11.428	11.504	13.952	13.362
20 <sup>th</sup> perc	15.735	15.171	12.203	12.286	14.938	14.482
30 <sup>th</sup> perc	15.968	15.620	12.693	12.781	15.654	15.243
40 <sup>th</sup> perc	16.146	15.940	13.082	13.174	16.158	15.871
50 <sup>th</sup> perc	16.298	16.227	13.398	13.493	16.591	16.384
60 <sup>th</sup> perc	16.431	16.476	13.685	13.782	16.990	16.879
70 <sup>th</sup> perc	16.581	16.731	13.949	14.049	17.392	17.367
80 <sup>th</sup> perc	16.739	17.018	14.208	14.310	17.824	17.903
90 <sup>th</sup> perc	16.957	17.422	14.474	14.579	18.323	18.580

Notes: 1. In all cases, discounting is carried out at the spot interest rate for the relevant maturity as determined by the calibrated CIR interest-rate model. 2. 'PC' means that the simulations assume the parameters of the stochastic mortality model to be certain. 3. 'PU' means that the simulations allow for uncertainty in the parameters of the Cairns-Blake-Dowd (CBD) stochastic mortality model using the method outlined in Cairns *et alia* (2006a). 4. Longevity risk is modelled using simulated values of  $\kappa_1(40)$  and  $\kappa_2(40)$  obtained using the CBD model and taking account of interim stochastic mortality improvement in the period since 2002. These parameter values are based on estimates of the mortality of English and Welsh males aged 65 over the period 1982-2002. See equations (1)-(2) in the text. 5. The instantaneous spot interest rate at  $T=40$  is assumed to be equal to 0.04. 6. The instantaneous spot interest rate and term structure of interest rates are assumed to be governed by a CIR process (given by equation (6) in text) with parameters  $\alpha = 0.20$ ,  $\sigma = 0.10$  and  $\bar{r} = 0.04$ . 7. Mortality rates are assumed to be unchanged from their current levels.

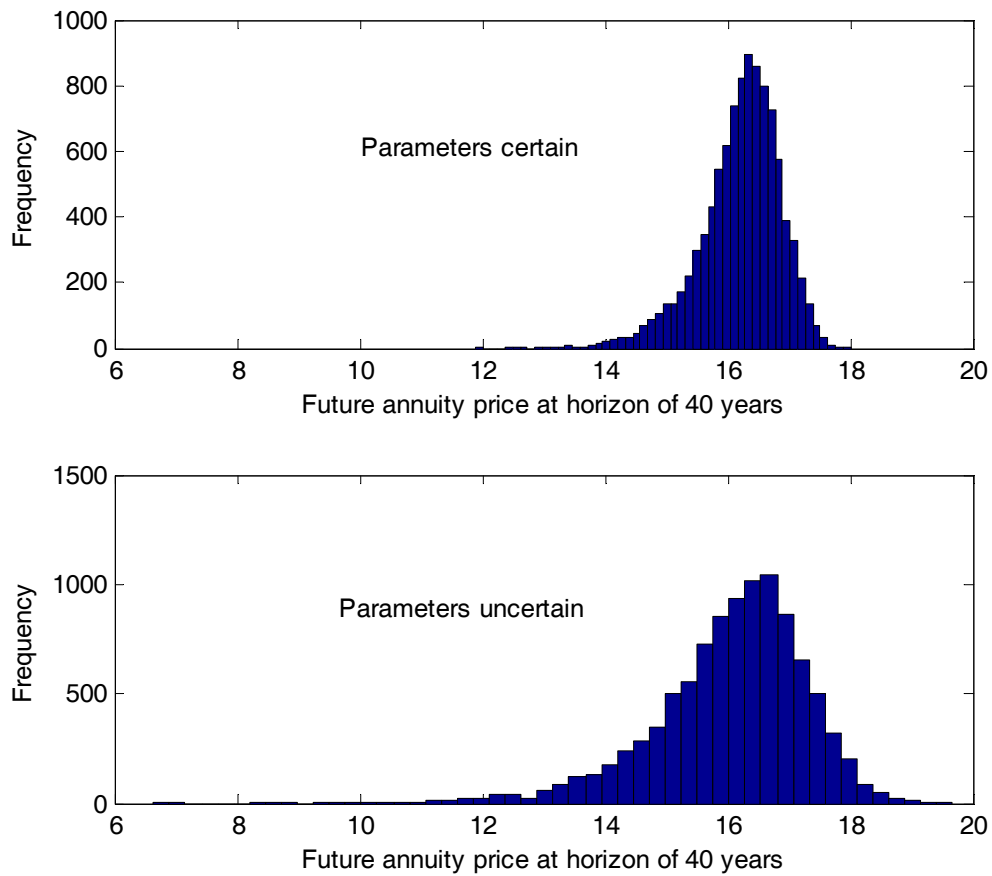
## FIGURES

**Figure 1: Density Function for Future CIR Instantaneous Spot Interest Rate at a 40-Year Horizon**



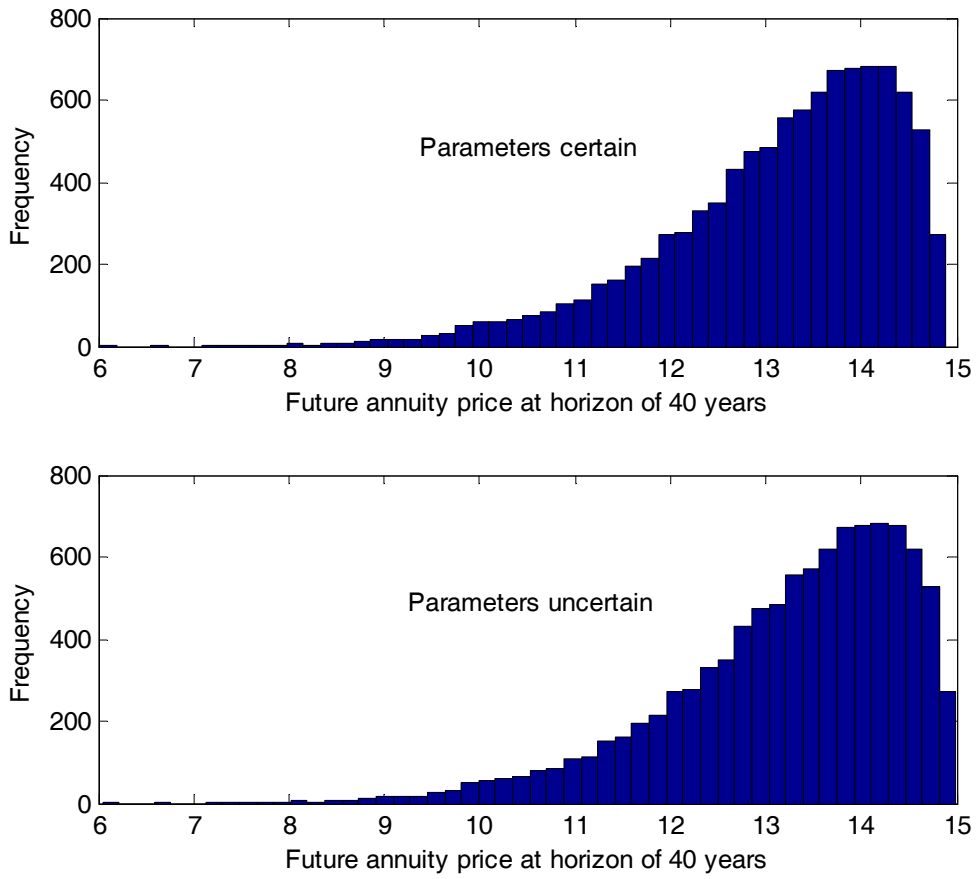
Notes: Figures shows a non-central chi-squared distribution based on (4) calibrated to  $\alpha = 0.20$ ,  $\sigma = 0.10$ ,  $\bar{r} = 0.04$ ,  $r(0) = 0.04$  and  $T = 40$ .

**Figure 2: Histogram of Simulated Future Annuity Values under Longevity Risk but no Interest-Rate Risk**



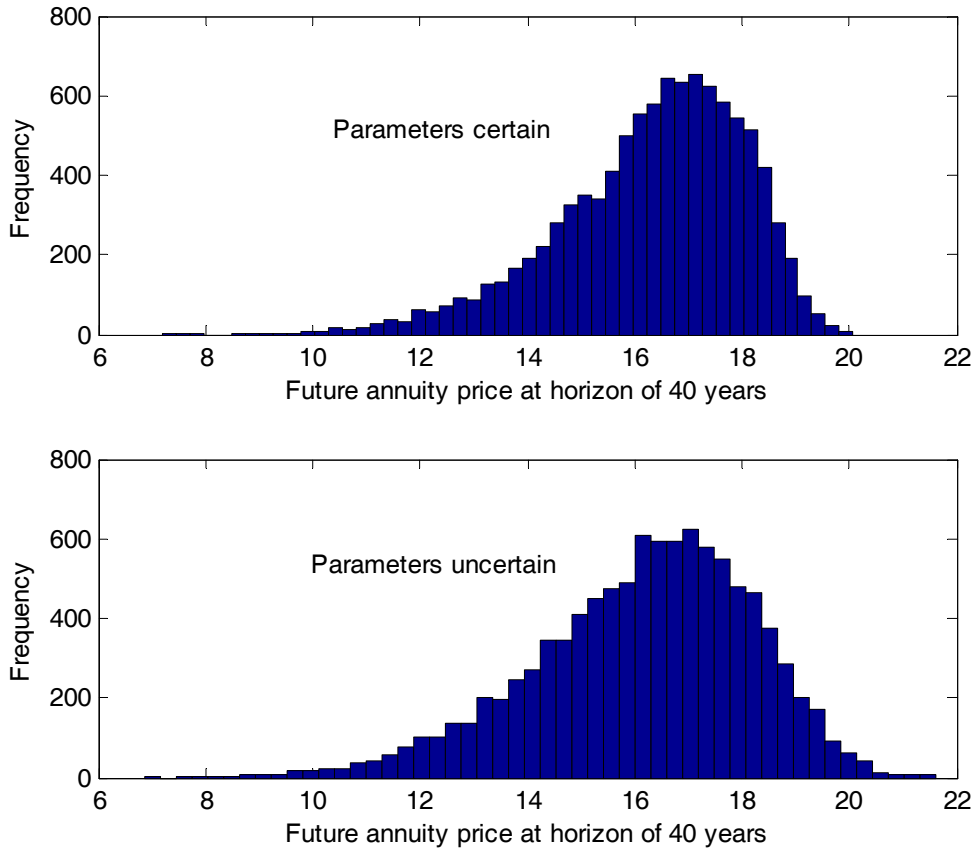
Notes: As per Notes 2, 3, 4 and 5 to Table 2.

**Figure 3: Histogram of Simulated Future Annuity Values under Interest-Rate Risk but no Longevity Risk**



Notes: As per Notes 2, 3, and 6 to Table 2.

**Figure 4: Histogram of Simulated Future Annuity Values under Longevity Risk and Interest-Rate Risk**



Notes: As per Notes 2, 3, and 4 to Table 2.