Mortality Bond Pricing

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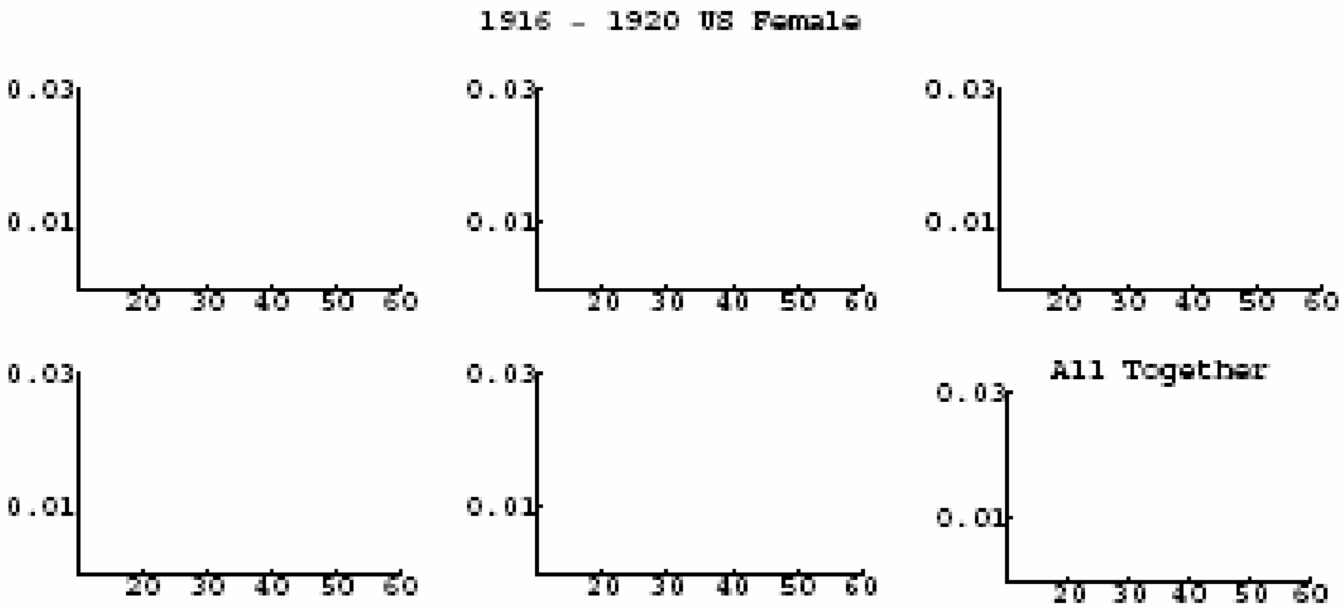
Outline

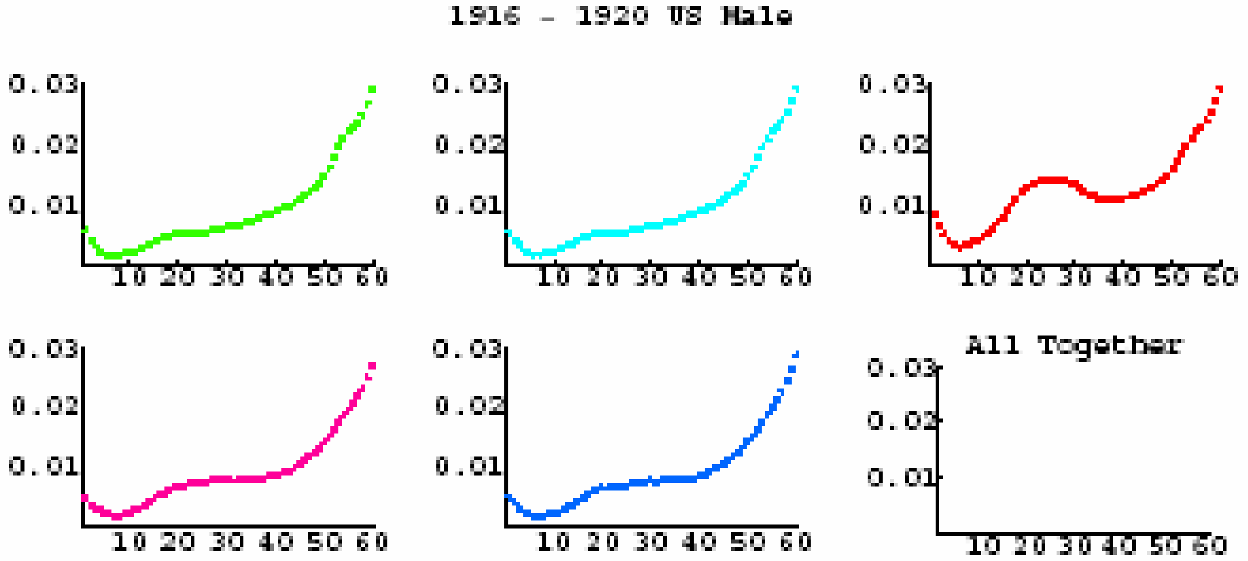
- Mortality empirical dynamics
- Mortality securities
  - Swiss Re bond
  - Hypothetical US-based mortality bond
- Mortality dynamic model
- Wang Transform
- Pricing mortality bonds

1916–1920 US Female Population q_x



Human Mortality Database. University of California, Berkeley (USA), and Max Planck Institute for Demographic Research (Germany). Available at www.mortality.org or www.humanmortality.de (Data downloaded November 1 – 10, 2004).

1916–1920 US Male Population q_x

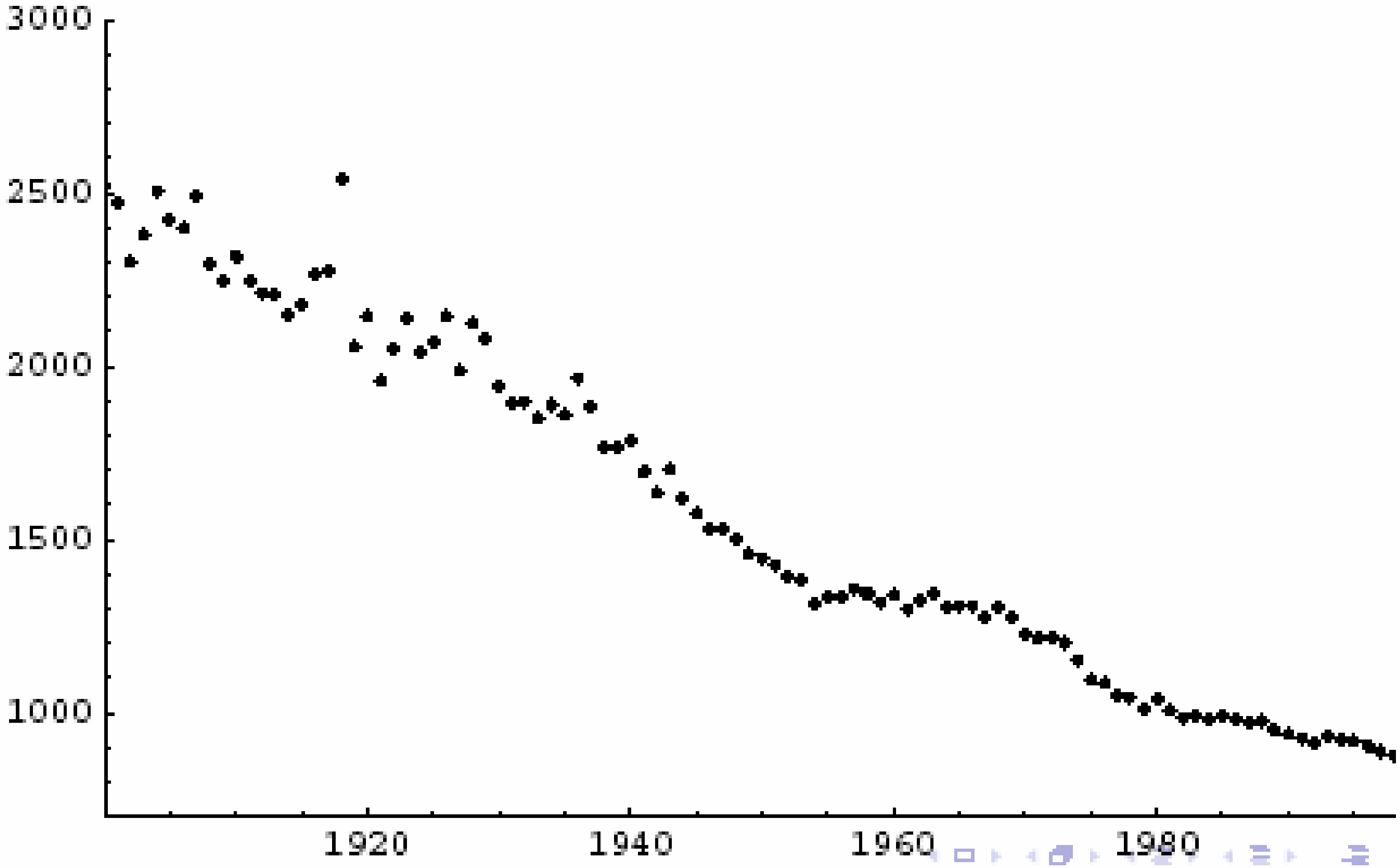


Swiss Re Deal

- Issued December 2003, matures January 1, 2007, a three year deal.
- No coupons at risk
- Priced to sell at par with a coupon of LIBOR + 1.35%.
- Principal at risk. q = weighted average population mortality in US, UK, France, Italy, and Switzerland. q_0 = 2002 level, $q_i = 2002 + i$ level, and $q = \max(q_1, q_2, q_3, q_4)$

$$\text{Maturity value} = \begin{cases} 400,000,000 & \text{if } q \leq 1.3q_0 \\ 400,000,000\dfrac{1.5q_0 - q}{0.2q_0} & \text{if } 1.3q_0 < q \leq 1.5q_0 \\ 0 & \text{if } q > 1.5q_0 \end{cases}$$

1900 – 1998 US Total Population Death Rate per 100,000



Index dynamics

- The mortality index, $q(t)$, will be continuous most of the time.
- If α, λ, k, and σ are constants, we can solve the differential equation (1)

$$q(t) = q(0)\exp\left[(\alpha - \frac{1}{2}\sigma^2 - \lambda k)t + \sigma W(t)\right] Y(N(t)), \quad (1)$$

- $Y(N(t)) = 1$ if $N(t) = 0$ and $Y(N(t)) = \prod_{j=1}^{N(t)} Y_j$ for $N(t) \geq 1$ where the size of the j^{th} jump, Y_j, is independently and identically log-normally distributed with parameters m and s.
- $\log q(t)|N(t) = n$ is normal which allows us to calculate the density numerically as a Poisson mixture of normal densities.

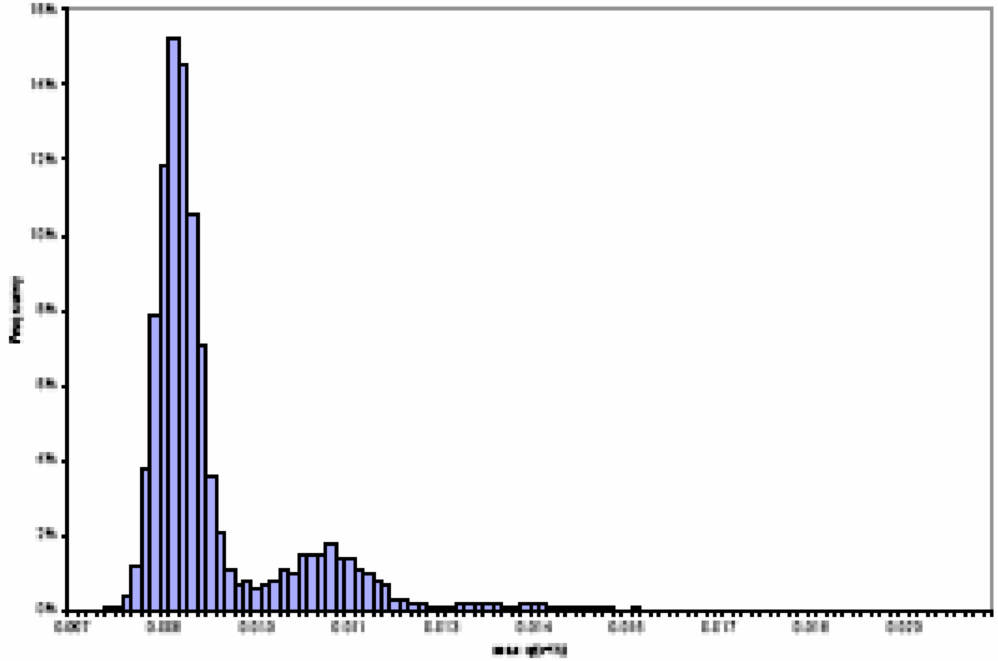
Maximum Likelihood Parameter Estimates

- US Population Mortality Index 1900 – 1998.
- Maximum likelihood estimates:

$$\alpha = -0.0100$$
$$\sigma = 0.0304$$
$$\lambda = 0.0456$$
$$m = -0.0266$$
$$s = 0.1096$$
$$k = -0.0204$$

- Rejects the model without jumps at significance level 0.1%
- In our simulation, we assumed that a $m = 15\%$. An increase of about 16.9% on average occurs once 1 in $1/\lambda = 22$ years. (and other parameters have the MLE values.)

Distribution of $q = \max(q_1, q_2, q_3, q_4.q_5)$



Mean	0.0092	Maximum	0.0204
Standard Deviation	0.0012	Minimum	0.0076

Measure of Risk-Return Performance

- For assets, the Sharpe ratio: $\lambda = \dfrac{\mathrm{E}(R) - r}{\sigma}$ is the excess return per unit of volatility.
- λ is also called "market price of risk."
- For insurance risks, the Sharpe ratio is the risk load per unit of volatility
- Limitation: insurance loss distributions are skewed

Sharpe Ratio for Long Time Horizons

- Sharpe Ratio λ increases with time horizon:

$$\lambda(T) = \lambda(1)T^b$$

where $0.5 \leq b \leq 1$ where T is the time horizon.
- $b = 0.5$ if reserve development follows a Brownian motion.
- For stocks, T is the holding period.
- For long-tailed liabilities, T is the average payment duration.

Actuarial World

- Ground-up Loss X has loss exceedence curve:

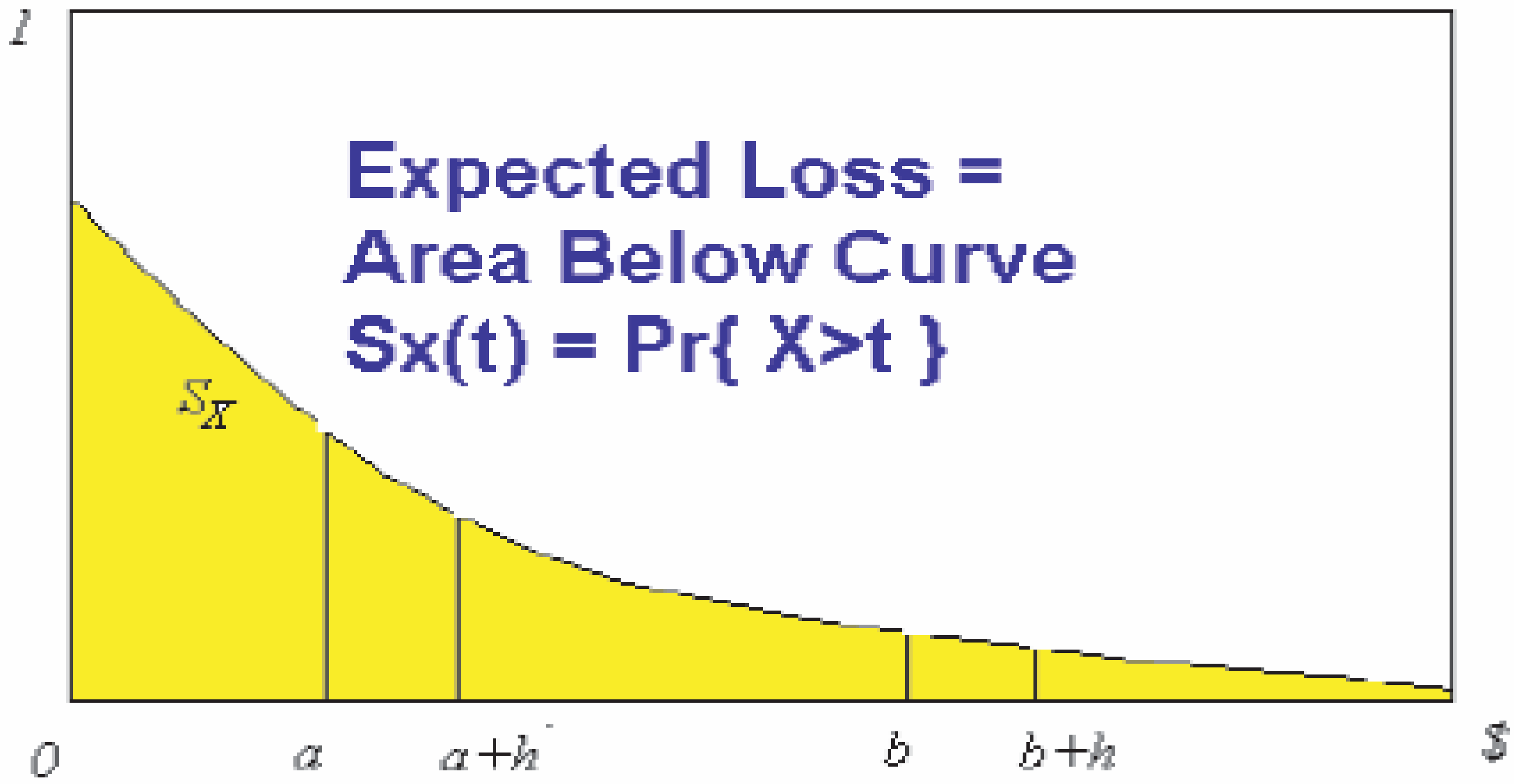
$$S_X(t) = 1 - F_X(t) = \Pr(X > t).$$

- Layer $X(a, a+h) = \min(X, a+h) - \min(X, a)$ where $a =$ retention and $h =$ limit.
-

$$\mathrm{E}(X) = \int_0^\infty S(t)\,\mathrm{d}t$$

$$\mathrm{E}[X(a, a+h)] = \int_a^{a+h} S(t)\,\mathrm{d}t$$

Loss Exceedence Curve

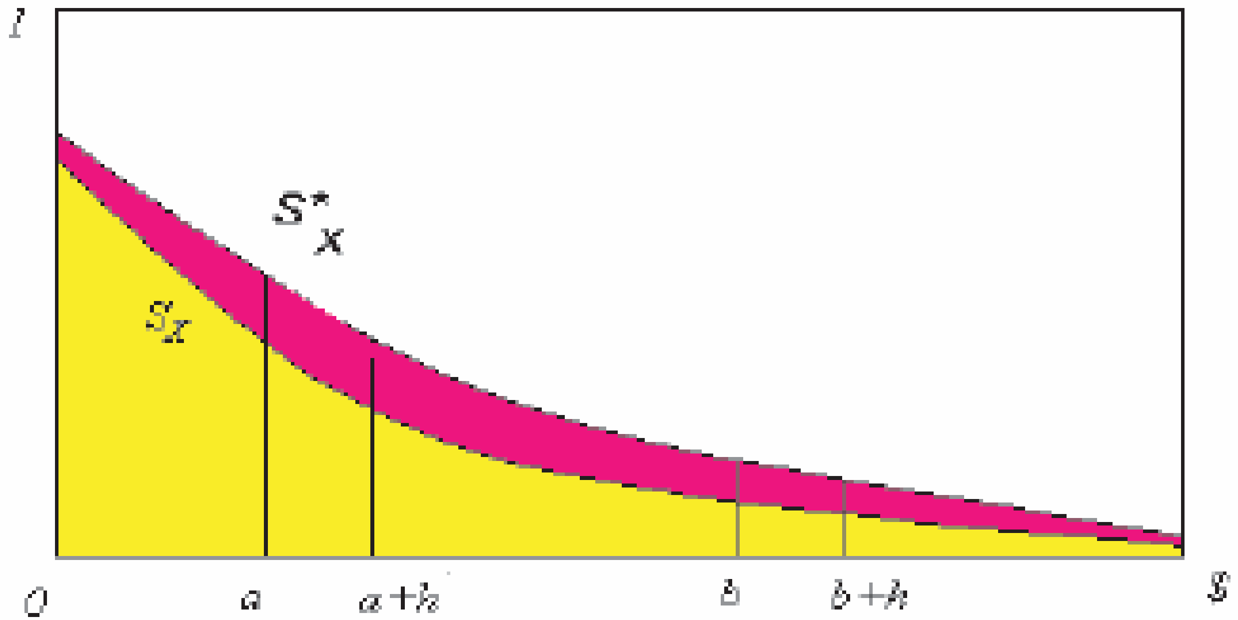


Insight of Gary Venter (ASTIN 1991)

Insurance prices by layer implies a transformed distribution

- layer $(t, t + dt)$ loss: $S_X(t)\,dt$
- layer $(t, t + dt)$ price: $S_X^*(t)\,dt$
- implied transform: $S_X(t) \to S_X^*(t)$

"Insurance prices by layer imply a transformed distribution."



Wang Transform

- Given a distribution with cdf $F(x)$, the Wang transform yields the new distribution

$$F^*(x) = \Phi[\Phi^{-1}(F(x)) - \lambda]$$

where $\Phi(z)$ is the standard normal cdf.

- Example: Suppose $F(x) = 0.95$ and $\lambda = 0.3$

$$\Phi^{-1}(F(x)) = \Phi^{-1}(0.95) = 1.645$$
$$\Phi^{-1}(F(x)) - \lambda = 1.645 - 0.3 = 1.345$$
$$F^*(x) = \Phi(1.345) = 0.911$$

- Fair Price is derived from the expected value under the transformed distribution $F^*(x)$.

Extending the Sharpe Ratio Concept

- If $F_X(x)$ is normal (μ, σ), then $F_X^*(x)$ is normal $(\mu + \lambda\sigma, \sigma)$.
- If $F_X(x)$ is lognormal (μ, σ), then $F_X^*(x)$ is lognormal $(\mu + \lambda\sigma, \sigma)$.
- The Wang transform recovers the CAPM and Black/Scholes [Wang, 2000].
- λ extends the Sharpe ratio to skewed distributions.

A Two-Factor Model

Wang transform with adjustment for parameter uncertainty:

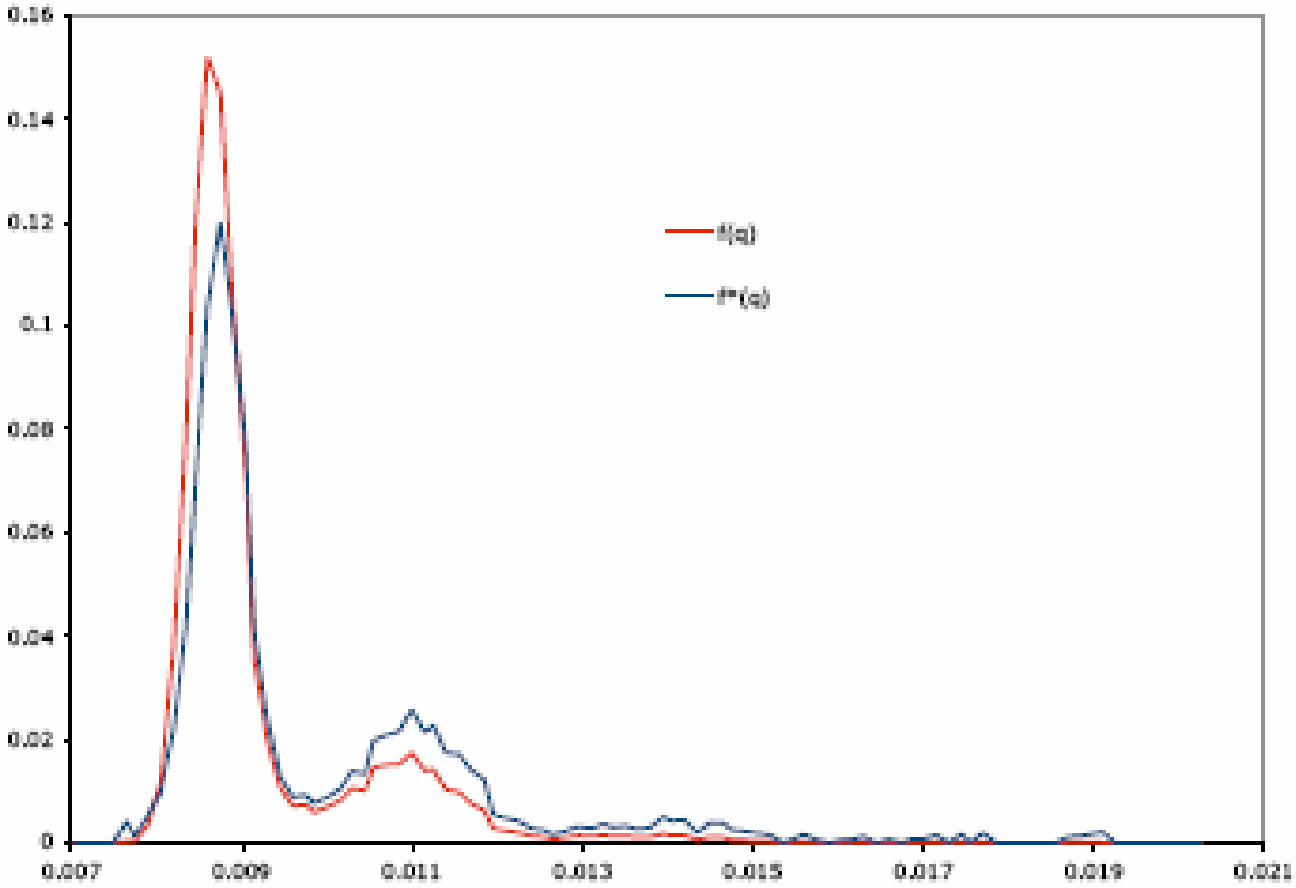
$$F_X^*(x) = Q[\Phi^{-1}(F_X(x)) - \lambda]$$

where

$\Phi(z)$ is the standard normal cdf

$Q(x)$ is the Student–t cdf with k degrees of freedom

Transformed Distribution of q



Prices – in terms of spread over risk free rate

Here are the yield spreads in excess of the risk free rate to make the price equal to the face amount for three values of λ and $k = 6$ degrees for freedom in each case.

λ	Par Spread	Risk
0.15	0.0118	Life insurance
0.30	0.0145	Annuity
0.45	0.0177	Cat bond

Conclusion

- Mortality Index Model: Jump–diffusion
  - US population data has jumps
  - MLE are tractable
- Wang Transform Pricing
  - The two factor transform is easy to use.
  - Our price for a US based "Swiss Re" style bond is close to the price quoted in the trade press.

Y. Lin and S. H. Cox. Securitization of mortality risks in life annuities. *Journal of Risk and Insurance*, 2005. (In press).

S. Wang. A class of distortion operations for pricing financial and insurance risks. *Journal of Risk and Insurance*, 67(1):15–36, 2000.