

Modelling the Mortality Term Structure

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Joint work with David Blake & Kevin Dowd

Paper: Pricing death: frameworks for the valuation and securitization of mortality risk (2004)

Updated version at: <http://www.ma.hw.ac.uk/~andrewc>

Aims of this work

Earlier today: EIB/BNP Paribas Longevity Bond

- How do we price this bond?
- In an arbitrage-free market
how might the price of this bond evolve through time?

We need:

- (a) a stochastic mortality model;
- (b) a stochastic interest-rate model.

Aims of this work

- There are many possible stochastic mortality models.
- What constitutes a helpful stochastic mortality model for pricing and fair valuation?
- Interest-rate theory \Rightarrow
 - ready-made frameworks for stochastic mortality
 - new stochastic mortality models
 - consistent **pricing** frameworks

PLAN FOR TALK

- Aims
- What is stochastic mortality?
- Why do we need to model stochastic mortality?
- Modelling: basic ingredients for arbitrage-free markets
- Modelling: different frameworks
- Criteria for a good model

STOCHASTIC MORTALITY

n lives, probability p of survival, N survivors

- Unsystematic mortality risk:

$$\Rightarrow N|p \sim \text{Binomial}(n, p)$$

$$\Rightarrow \text{risk is diversifiable, } N/n \longrightarrow p \quad \text{as } n \longrightarrow \infty$$

- Systematic mortality risk:

$$\Rightarrow p \text{ is uncertain}$$

$$\Rightarrow \text{risk associated with } p \text{ is not diversifiable}$$

STOCHASTIC MODELS FOR MORTALITY RISK

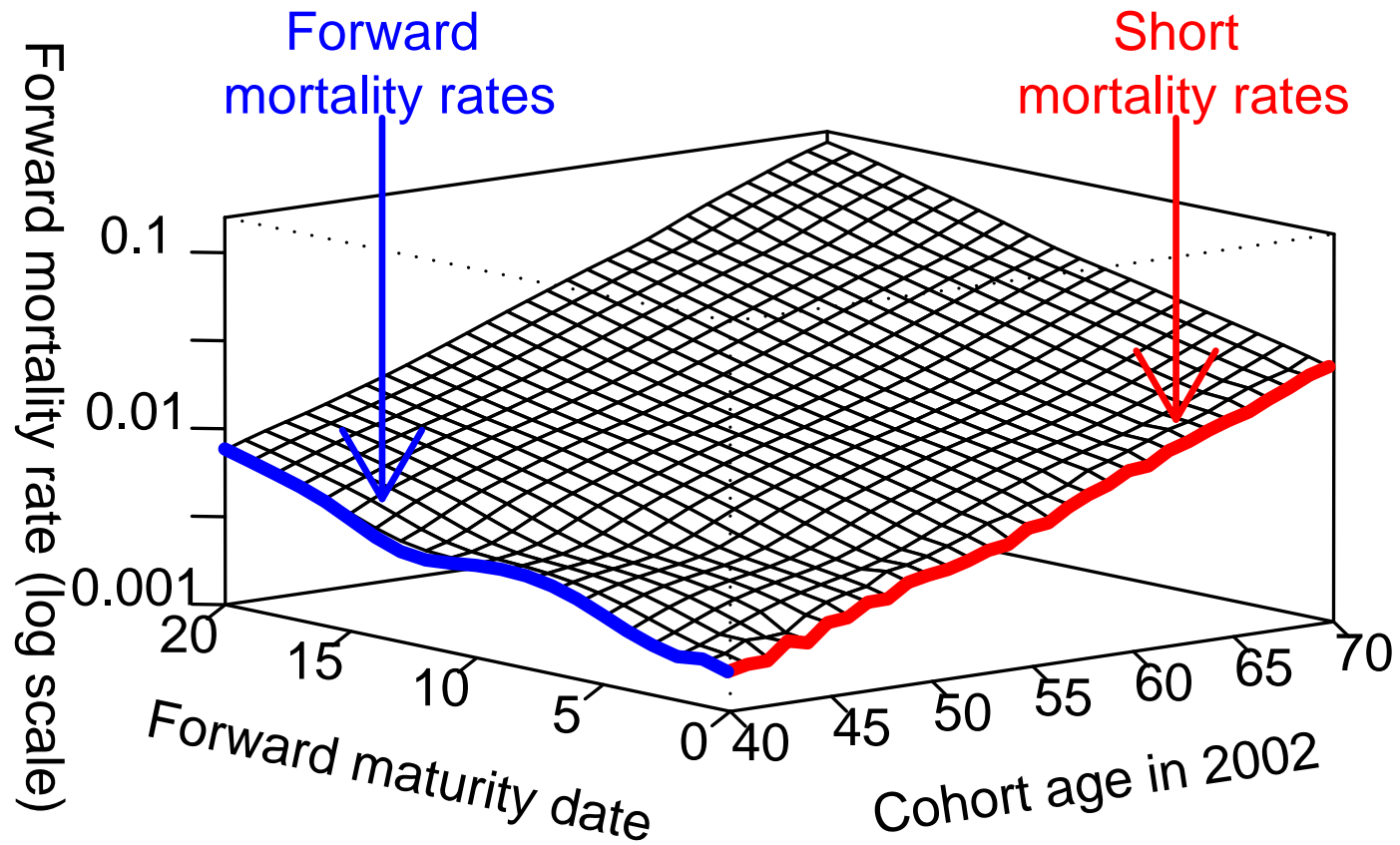
Why do we need a stochastic model for mortality?

- To calculate quantile reserves (VaR)
- To calculate fair values
 - especially contracts with embedded options
- To price mortality-linked securities

MORTALITY-LINKED SECURITIES

- Short-term **catastrophe bonds** (Swiss Re, Dec. 2003)
- Long-term **longevity bonds** (EIB/BNP, Nov. 2004)
 - cashflows linked to survivorship index
 - (Short report available later this week.)
- **Survivor swaps** (some OTC contracts???)
 - swap fixed for mortality-linked cashflows
- **Annuity futures**
 - traded contract; underlying=market annuity rates; many exercise dates

The term-structure of mortality



STOCHASTIC MODELLING

$\mu(t, x)$ = force of mortality at t for individuals aged x

$r(t)$ = risk-free rate of interest

$r(t), \mu(t, x)$ represent very different quantities

Mathematically we can treat $r(t), \mu(t, x)$ as equivalent

Cash account, $C(t)$:

$$\begin{aligned} C(t)^{-1} &= \exp \left[- \int_0^t r(s) ds \right] \\ &= \text{risk-free discount factor} \end{aligned}$$

Survivor Index:

$$\begin{aligned} S(u, y) &= \exp \left[- \int_0^u \mu(t, y + t) dt \right] \\ &= \text{Prob. of survival of } (y) \text{ from time 0 to time } u \\ &\quad \text{given knowledge of evolution of } \mu(t, x) \end{aligned}$$

FUNDAMENTAL SECURITIES

1. Fixed-interest zero-coupon bonds

$$P(t, T) = \text{Price at } t \text{ for } \$1 \text{ at time } T$$

2. Zero-coupon survivor (longevity) bond

$$\tilde{B}(t, T, x) = \text{Price at } t \text{ for } \$ S(T, x) \text{ at time } T$$

Approximately: BNP Paribas = $\sum_{T=1}^{25} \tilde{B}(t, T, 65)$

What can we learn from interest-rate modelling?

RISK-NEUTRAL PRICING

We postulate the existence of a risk-neutral pricing measure Q .

$$P(t, T) = E_Q \left[e^{-\int_t^T r(s) ds} \mid \mathcal{H}_t \right]$$

$$\tilde{B}(t, T, x) = E_Q \left[e^{-\int_t^T r(s) ds} S(T, x) \mid \mathcal{H}_t \right]$$

Pricing under $Q \Rightarrow$ dynamics under P are arbitrage free

NO requirement for liquidity, or zero transaction costs

Assumption: $\mu(t, y)$ is independent of $r(t)$

- Not okay for Swiss Re catastrophe bond
- Reasonable assumption for longevity bonds

$$\Rightarrow \tilde{B}(t, T, x) = E_Q \left[e^{-\int_t^T r(s) ds} \mid \mathcal{F}_t \right] E_Q [S(T, x) \mid \mathcal{M}_t]$$

$$= P(t, T) B(t, T, x)$$

$$B(t, T, x) = E_Q [S(T, x) \mid \mathcal{M}_t]$$

$$\begin{aligned} \frac{B(t, T, x)}{B(t, t, x)} &= \frac{\tilde{B}(t, T, x)}{P(t, T)B(t, t, x)} \\ &= E_Q \left[e^{-\int_t^T \mu(s, x+s) ds} \mid \mathcal{M}_t \right] \\ &= \text{risk-neutral probability at } t \text{ that } (x + t) \\ &\quad \text{survives from time } t \text{ to time } T \\ &= \text{spot survival probability, } p_Q(t, T, x) \end{aligned}$$

TYPES OF STOCHASTIC MORTALITY MODEL

We can use the same **frameworks** as interest-rate

modelling: $\mu(t, x + t)$ is equivalent to $r(t)$

(but we might not use the same **models!!!**)

- **Short-rate modelling framework** (e.g. CIR)
- **Forward-rate modelling framework** (e.g. HJM)
- **Positive-interest framework** (e.g. Flesaker-Hughston)
- **Market Models** (e.g. BGM)

Key point

Stochastic mortality models don't have to match an interest-rate model.

... but it helps with the pricing process.

SHORT-RATE MODELLING FRAMEWORK

e.g. one-factor models

$$d\mu(t, y) = a(t, y)dt + b(t, y)d\tilde{W}(t)$$

Examples:

- Lee & Carter (1992) and followers (discrete time)
- Cairns, Blake, Dawson and Dowd (2005) (discrete time) model for assessing risk in longevity bond
- Milevsky & Promislow (2001), Dahl (2004) (cont. time)

FORWARD-RATE MODELS

- Begin with spot survival probabilities:

$$p_Q(t, T, x) = E_Q \left[e^{-\int_t^T \mu(s, x+s) ds} \mid \mathcal{M}_t \right]$$

for $T = t + 1, t + 2, \dots$ and current ages

$$x + t = 20, \dots, 90$$

- Framework \Rightarrow constraints on how dynamics of the $p_Q(t, T, x)$ interact
- **Smith and Olivier** (slides at <http://www.actuaries.ie>)

MARKET MODELS

Interest rates:

- Forward swap rates are log-normal (Jamshidian, 1997)
- Forward LIBOR rates are L-N (Brace-Gatarek-Musiela, 1997)

Mortality (Cairns, Blake and Dowd, 2004):

- Forward life annuity rates are L-N
- Forward Survivor Credit Offer Rates (SCOR) are L-N

POSSIBLE CRITERIA FOR STOCHASTIC MORTALITY MODELS

- $\mu(t, x) > 0$ for all t, x
- Model consistent with historical data
- Future dynamics should be *biologically* reasonable
- Complexity of model appropriate for task in hand
- Model allows fast numerical computation
- Avoid mean reversion

Summary

- Range of frameworks possible for stochastic mortality models
- No one framework is intellectually superior to the rest
- Next steps:
 - Develop specific models based on these frameworks
 - Fit each model to historical data
 - Compare different models for quality of fit