International Association for the
Study of Insurance Economics

# Études et Dossiers

Extract from

## Études et Dossiers No. 317

**2nd International Longevity Risk and Capital Market Solutions Symposium**

**24 April 2006, Chicago**

August 2006

Working Paper Series of
The Geneva Association



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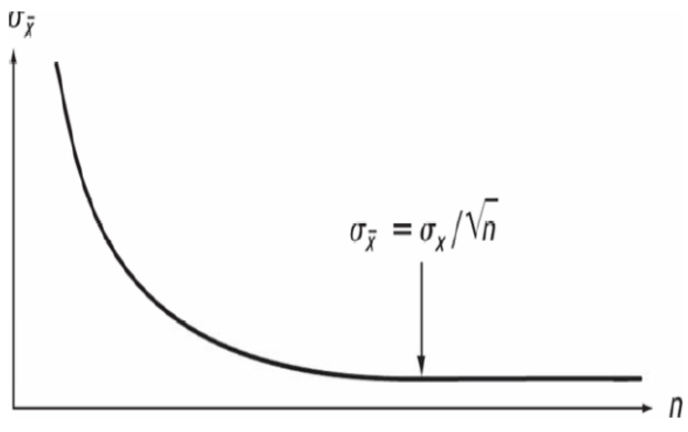
Layout & Distribution: Valeria Kozakova

# Multivariate Exponential Tilting and Pricing Implications for Mortality Securitization

Sameul H. Cox, Yijia Lin and Shaun Wang

References

## The Beauty of Risk Pooling

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## The Curse of Risk Pooling

- Law of Large Numbers

$$\mathrm{Var}(\bar{T}) = \frac{1}{n}\mathrm{Var}(T|\theta) \to 0. \tag{1}$$

- Events that cause simultaneous effect

$$\begin{aligned} \mathrm{Var}(\bar{T}) &= \mathrm{E}[\mathrm{Var}(\bar{T}|\theta)] + \mathrm{Var}\mathrm{E}(\bar{T}|\theta) \\ &= \mathrm{E}[\frac{1}{n}\mathrm{Var}(T|\theta)] + \mathrm{Var}[\mathrm{E}(T|\theta)] \\ &\to \mathrm{Var}[\mathrm{E}(T|\theta)] \neq 0. \end{aligned} \tag{2}$$

  - Pooling technique breaks down.

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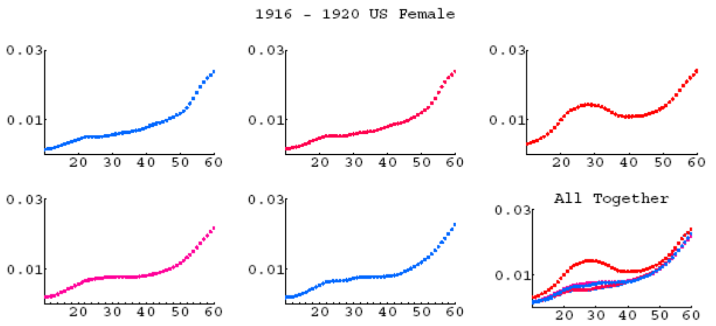
References

## Shocks to a Life Insurer

- 1916–1920 US Female Population $q_x$

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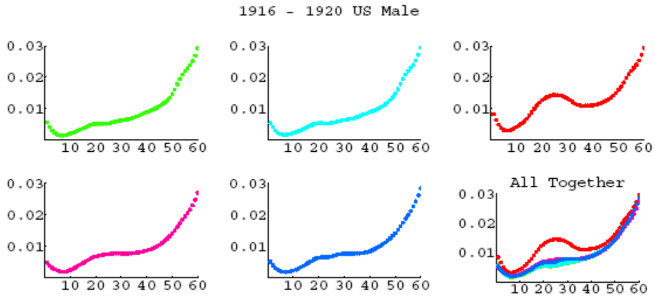
## Shocks to a Life Insurer (Cont')

- 1916–1920 US Male Population $q_x$

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## Shocks to a Life Insurer (Cont')

December 2004 Earthquake and Tsunami Death Toll and Percentage Excess Death Rates by Country

| Country | Confirmed deaths[a] | Missing[a] | % Excess Death Rate[b] |
|---|---|---|---|
| Indonesia | 127,420 | 116,368 | 16.58% |
| Sri Lanka | 38,195 | 4,924 | 33.81% |
| India | 10,779 | 5,614 | 0.18% |
| Thailand | 5,395 | 2,991 | 1.90% |
| Somalia | 298 | - | 0.21% |
| Myanmar | 90 | - | - |
| Maldives | 82 | - | 3.25% |
| Malaysia | 68 | - | 0.06% |
| Tanzania | 10 | - | - |
| Bangladesh | 2 | - | - |
| Kenya | 1 | - | - |
| Total | 182,340 | 129,897 | - |

[a]Source: *Associated Press* on March 4, 2005; [b]Based on the authors' calculation.

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References

## Shocks to an Annuity Insurer/Pension Plan

- Dramatic mortality improvement
- Increased demand for annuities (Mitchell et al., 2001)
- Private DB pension plans
- Existing social security system in the US

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References

## Life Insurance-linked Cat Securitization

- Multiple risks
  - Interest rate risk
  - Policyholder lapse risk
  - Regulatory risk
  - Mortality risk
  - Insurer policy dividend decisions
- **Pure mortality or longevity risk**
  - Increase transparency (Cowley and Cummins, 2005)
  - Diversification benefit (Lin and Cox, 2005)

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## Swiss Re Deal

- Issued December 2003, matures January 1, 2007, a three year deal.
- No coupons at risk
- Priced to sell at par with a coupon of LIBOR + 1.35%.
- Principal at risk. $q$ = weighted average population mortality in US (70%), UK (15%), France (7.5%), Italy (5%), and Switzerland (2.5%). $q_0$ = 2002 level and $q = \max(q_{2004}, q_{2005}, q_{2006})$

$$\text{Maturity value} = \begin{cases} 400,000,000 & \text{if } q \leq 1.3q_0 \\ 400,000,000\dfrac{1.5q_0 - q}{0.2q_0} & \text{if } 1.3q_0 < q \leq 1.5q_0 \\ 0 & \text{if } q > 1.5q_0 \end{cases}$$



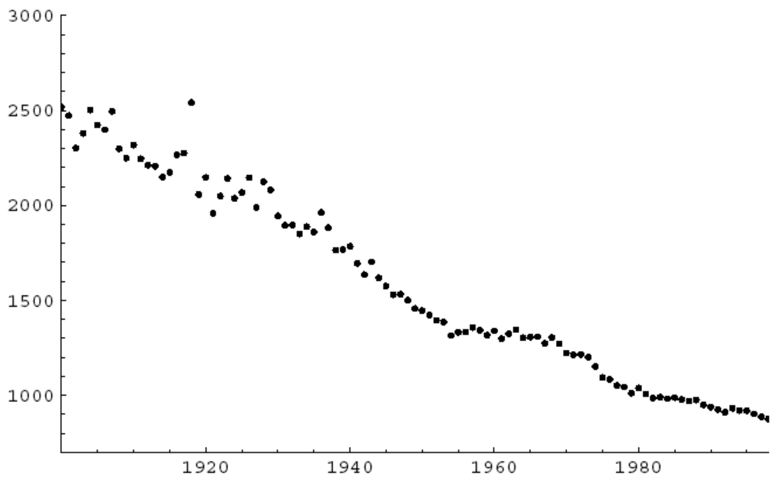


## Mortality Securitization Modeling Literature

- Existing mortality securitization pricing theory
  - Ignore mortality jumps
    - Itô–type stochastic process (Dahl, 2003; Milevsky and Promislow, 2001; Cairns et al., 2004)
    - Econometric methods (Renshaw et al., 1996; Sithole et al., 2000; Lee and Carter, 1992; Lee, 2000)

References

## 1900 – 1998 US Population Death Rate per 100,000

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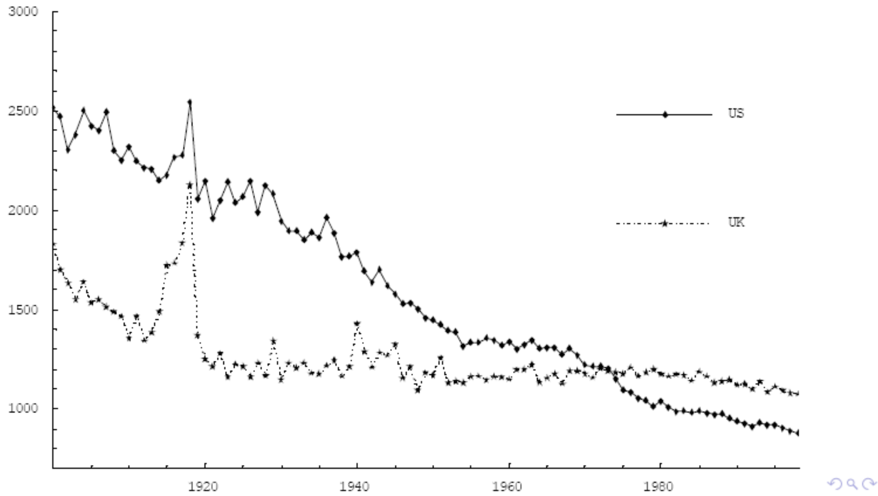
## 1900 – 1998 US and UK Population Death Rate per 100,000

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    - Econometric methods (Renshaw et al., 1996; Sithole et al., 2000; Lee and Carter, 1992; Lee, 2000)
  - Ignore mortality correlation
- Improve existing mortality securitization models
  - Pricing the Swiss Re bond by multivariate exponential tilting with jump processes

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## Multivariate Exponential Tilting

Consider $n$ variables $X_1, X_2, ..., X_n$ and $k$ references $Y_1, Y_2, ..., Y_k$ on a probability space $(\Omega, P)$.

**Definition 1.** For each scenario $\omega$ in the probability space $(\Omega, P)$, the exponential tilting of $X_1, X_2, ..., X_n$ with respect to references $Y_1, Y_2, ..., Y_k$ is defined by the following p.d.f.:

$$\frac{f^*(x_1(\omega), x_2(\omega), ..., x_n(\omega))}{f(x_1(\omega), x_2(\omega), ..., x_n(\omega))} = c\left[\exp\left(\Sigma_{j=1}^{k} \lambda_j Y_j(\omega)\right)\right], \quad (3)$$

where $\lambda_1, \lambda_2, ..., \lambda_k$ are real-valued parameters that control the magnitude of risk-adjustment, and $c$ is a normalizing coefficient.





## Normalized Multivariate Exponential Tilting

**Definition 2.** Assume that there exist standard normal variables $Z_1, Z_2, ..., Z_k$ such that

$$Y_1 = F_{Y_1}^{-1}(\Phi(Z_1)), Y_2 = F_{Y_2}^{-1}(\Phi(Z_2)), ..., Y_k = F_{Y_k}^{-1}(\Phi(Z_k)). \quad (4)$$

Equation (4) can be obtained by percentile mapping.

$$\frac{f^*(x_1, x_2, ..., x_n)}{f(x_1, x_2, ..., x_n)} = c\mathrm{E}\left[\exp\left(\Sigma_{j=1}^{k} \lambda_j Z_j\right) | X_1 = x_1, X_2 = x_2, ..., X_n = x_n\right]. \quad (5)$$

## Correlation Matrix

**Theorem 1.**

Assume that $\{X_1, X_2, ..., X_n; Y_1, Y_2, ..., Y_k\}$ follow a normal copula with correlation matrix:

$$\Sigma = \begin{pmatrix} \Sigma_{xx} & \Sigma_{xy} \\ \Sigma_{xy} & \Sigma_{yy} \end{pmatrix}. \qquad (6)$$





## Market Price of Risk

**Theorem 1. (Cont'd)**

The multivariate exponential tilting of $\{X_1, ..., X_n\}$ with respect to themselves $(Y_j = X_j, j = 1, ..., n)$ is equivalent to the multivariate Wang transform (Wang, 2006) with

$$F^*_{X_i}(x_i) = \Phi[\Phi^{-1}(F_{X_i}(x_i)) + \beta_i],$$

$$\text{and } \beta_i = \sum_{j=1}^{n} \rho_{X_i,Y_j} \cdot \lambda_j, (\text{ for } i = 1, 2, ..., n). \qquad (7)$$

The correlation matrix between $X_1, X_2, ..., X_n$ is unchanged after the normalized multivariate exponential tilting, $\Sigma^*_{xx} = \Sigma_{xx}$. Kijima (2006) reaches the same conclusion as equation (7) by using a multi-period equilibrium argument.

## US and UK Population Index Dynamics

$$q^{us}_{t+h}|\mathcal{F}_t = q^{us}_t \exp\left[(\alpha^{us} - \frac{1}{2}\sigma^{us\,2} - \Lambda^{us}k^{us} - \Lambda^{intl}k^{intl})h + \sigma^{us}\Delta W^{us}_t\right] \cdot \prod_{j>N^{us}_t}^{N^{us}_{t+h}} Y^{us}_j \prod_{i>N^{intl}_t}^{N^{intl}_{t+h}} Y^{intl}_i, \text{ and}$$

$$q^{uk}_{t+h}|\mathcal{F}_t = q^{uk}_t \exp\left[(\alpha^{uk} - \frac{1}{2}\sigma^{uk\,2} - \Lambda^{uk}k^{uk} - \Lambda^{intl}k^{intl})h + \sigma^{uk}\Delta W^{uk}_t\right] \cdot \prod_{j>N^{uk}_t}^{N^{uk}_{t+h}} Y^{uk}_j \prod_{i>N^{intl}_t}^{N^{intl}_{t+h}} Y^{intl}_i, \text{ where } \underline{\underline{\mathrm{Cov}(W^{us}_t, W^{uk}_t) = \rho\sigma^{us}\sigma^{uk}}}.$$



References

## Maximum Likelihood Parameter Estimates

- US and UK population mortality index 1900 – 1998.
- Maximum likelihood estimates:

| | | | | | |
|---|---|---|---|---|---|
| $\alpha^{us}$ | -0.0100 | $\alpha^{uk}$ | -0.0033 | | |
| $\sigma^{us}$ | 0.0308 | $\sigma^{uk}$ | 0.0237 | $\rho$ | 0.5299 |
| $\Lambda^{us}$ | $10^{-6}$ | $\Lambda^{uk}$ | 0.8533 | $\Lambda^{intl}$ | 0.0309 |
| $m^{us}$ | -0.0050 | $m^{uk}$ | -0.0114 | $m^{intl}$ | -0.0295 |
| $s^{us}$ | $10^{-6}$ | $s^{uk}$ | 0.0600 | $s^{intl}$ | 0.1412 |
| $k^{us}$ | -0.0050 | $k^{uk}$ | -0.0096 | $k^{intl}$ | -0.0193 |

Table: Maximum Likelihood Parameter Estimates Based on the US and UK Population Mortality Index 1900 –1998. The model without jumps is rejected at the significance level of 0.1%.

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## Correlation Matrix of US and UK Population Mortality Indices

- The correlation between $W^{us}$ and $W^{uk}$ is $\rho$.
- The jump sizes $Y^{us}$, $Y^{uk}$ and $Y^{intl}$ are independent of each other and of $W^{us}$ and $W^{uk}$.
- Assume that $\{W^{us}, W^{uk}, Y^{us}, Y^{uk}, Y^{intl}\}$ use themselves as references.

According to **Theorem 1**,

$$\Sigma^* = \Sigma = \begin{pmatrix} 1 & \rho & 0 & 0 & 0 \\ \rho & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0.5299 & 0 & 0 & 0 \\ 0.5299 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{8}$$





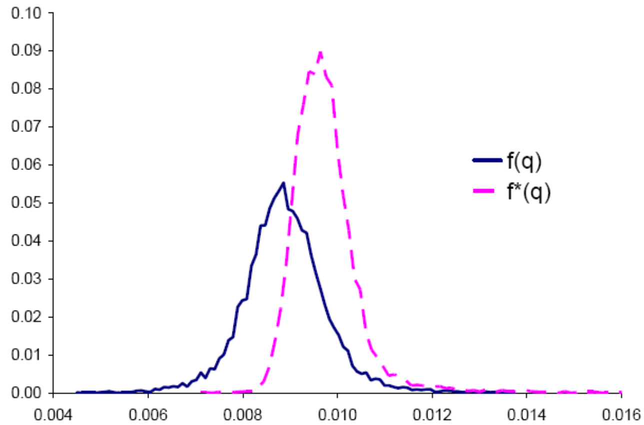
## Market Price of Risk of Swiss Re Bond

$$\begin{pmatrix} \beta_{W_{us}} \\ \beta_{W_{uk}} \\ \beta_{Y^{us}} \\ \beta_{Y^{uk}} \\ \beta_{Y^{intl}} \end{pmatrix} = \begin{pmatrix} 1 & \rho & 0 & 0 & 0 \\ \rho & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda_{W_{us}} \\ \lambda_{W_{uk}} \\ \lambda_{Y^{us}} \\ \lambda_{Y^{uk}} \\ \lambda_{Y^{intl}} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \rho & 0 & 0 & 0 \\ \rho & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda \\ \lambda \\ \lambda \\ \lambda \\ \lambda \end{pmatrix} = \begin{pmatrix} \lambda + \lambda\rho \\ \lambda + \lambda\rho \\ \lambda \\ \lambda \\ \lambda \end{pmatrix}. \tag{9}$$

References

## Transformed Distribution of $q = \max(q_{2004}, q_{2005}, q_{2006})$




Assume 80% on the US population mortality index and 20% on the UK Population mortality index.

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## Market Price of Risk of the Swiss Re Bond

| | $\beta_Y$ | $\beta_W$ | Par Spread |
|---|---|---|---|
| Swiss Re Bond | **0.83** | **1.27** | 1.35% |

- Market price of risk for the property-linked catastrophe bond $\lambda = \mathbf{0.45}$ with $k = 6$ degrees of freedom.
- The Swiss Re overcompensates the investors for their taking its mortality risks.
  - Minton et al. (2004) conclude that securitization of financial institutions is a contracting innovation aimed at lowering financial distress costs.
  - MorganStanley (2003) concludes that "Swiss Re must be taking a view that the cost of capital that is relieved via this transaction exceeds the effective net cost of servicing the bond".
  - To develop mortality securitization markets

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## Conclusions

- Insurers develop mortality securitization market to manage catastrophic mortality risks.
- Improve existing mortality securitization models
  - Include jump process
  - Use multivariate exponential tilting
    - Correlation
    - Incomplete market pricing method
- Swiss Re bond seems a good deal for investors.

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