Multivariate Exponential Tilting and Pricing Implications for Mortality Securitization

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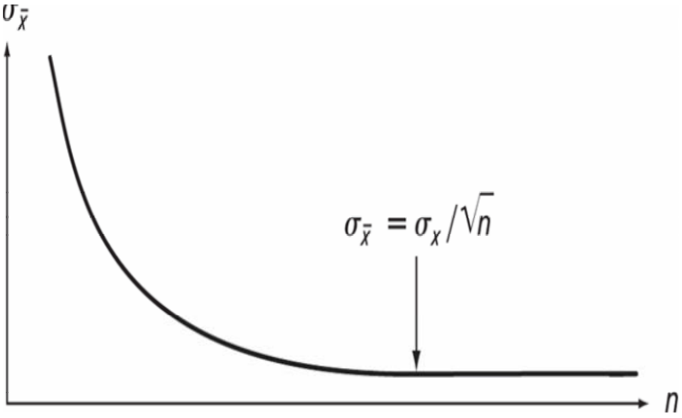
The Beauty of Risk Pooling



The Curse of Risk Pooling

- Law of Large Numbers

$$\mathrm{Var}(\bar{T}) = \frac{1}{n}\mathrm{Var}(T|\theta) \to 0. \tag{1}$$

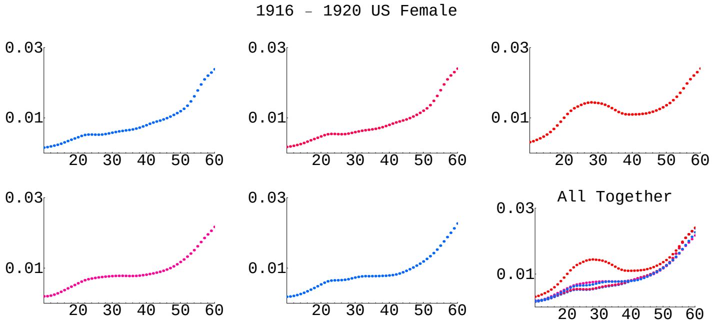
- Events that cause simultaneous effect

$$\begin{aligned}\mathrm{Var}(\bar{T}) &= \mathrm{E}[\mathrm{Var}(\bar{T}|\theta)] + \mathrm{Var}\mathrm{E}(\bar{T}|\theta) \\ &= \mathrm{E}[\frac{1}{n}\mathrm{Var}(T|\theta)] + \mathrm{Var}[\mathrm{E}(T|\theta)] \\ &\to \mathrm{Var}[\mathrm{E}(T|\theta)] \neq 0.\end{aligned} \tag{2}$$

  - Pooling technique breaks down.

Shocks to a Life Insurer

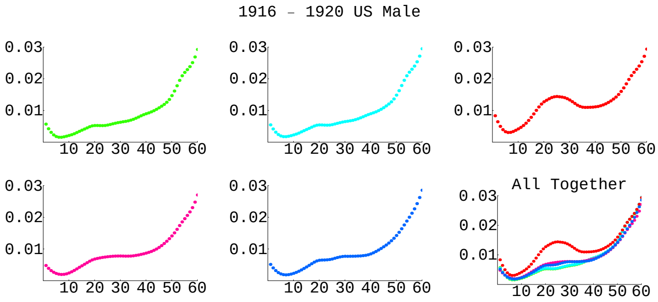
- 1916–1920 US Female Population q_x



Human Mortality Database. University of California, Berkeley (USA), and Max Planck Institute for Demographic Research (Germany). Available at www.mortality.org or www.humanmortality.de (Data downloaded November 1 – 10, 2004).

Shocks to a Life Insurer (Cont')

- 1916–1920 US Male Population q_x



Shocks to a Life Insurer (Cont')

December 2004 Earthquake and Tsunami Death Toll and Percentage Excess Death Rates by Country

Country	Confirmed deaths[a]	Missing[a]	% Excess Death Rate[b]
Indonesia	127,420	116,368	16.58%
Sri Lanka	38,195	4,924	33.81%
India	10,779	5,614	0.18%
Thailand	5,395	2,991	1.90%
Somalia	298	-	0.21%
Myanmar	90	-	-
Maldives	82	-	3.25%
Malaysia	68	-	0.06%
Tanzania	10	-	-
Bangladesh	2	-	-
Kenya	1	-	-
Total	182,340	129,897	-

[a]Source: *Associated Press* on March 4, 2005; [b]Based on the authors' calculation.

Shocks to an Annuity Insurer/Pension Plan

- Dramatic mortality improvement
- Increased demand for annuities (Mitchell et al., 2001)
- Private DB pension plans
- Existing social security system in the US

Life Insurance-linked Cat Securitization

- Multiple risks
  - Interest rate risk
  - Policyholder lapse risk
  - Regulatory risk
  - Mortality risk
  - Insurer policy dividend decisions
- **Pure mortality or longevity risk**
  - Increase transparency (Cowley and Cummins, 2005)
  - Diversification benefit (Lin and Cox, 2005)

Swiss Re Deal

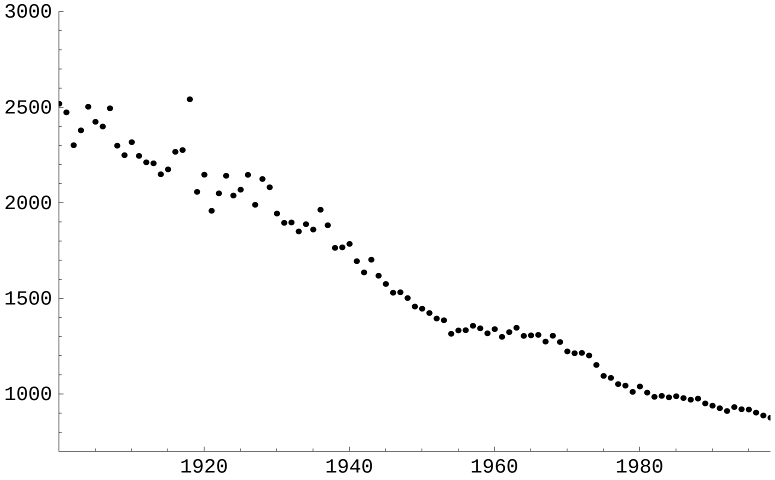
- Issued December 2003, matures January 1, 2007, a three year deal.
- No coupons at risk
- Priced to sell at par with a coupon of LIBOR $+ 1.35\%$.
- Principal at risk. $q =$ weighted average population mortality in US (70%), UK (15%), France (7.5%), Italy (5%), and Switzerland (2.5%). $q_0 =$ 2002 level and $q = \max(q_{2004}, q_{2005}, q_{2006})$

$$\text{Maturity value} = \begin{cases} 400,000,000 & \text{if } q \leq 1.3q_0 \\ 400,000,000\dfrac{1.5q_0 - q}{0.2q_0} & \text{if } 1.3q_0 < q \leq 1.5q_0 \\ 0 & \text{if } q > 1.5q_0 \end{cases}$$

Mortality Securitization Modeling Literature

- Existing mortality securitization pricing theory
  - Ignore mortality jumps
    - Itô–type stochastic process (Dahl, 2003; Milevsky and Promislow, 2001; Cairns et al., 2004)
    - Econometric methods (Renshaw et al., 1996; Sithole et al., 2000; Lee and Carter, 1992; Lee, 2000)

1900 – 1998 US Population Death Rate per 100,000



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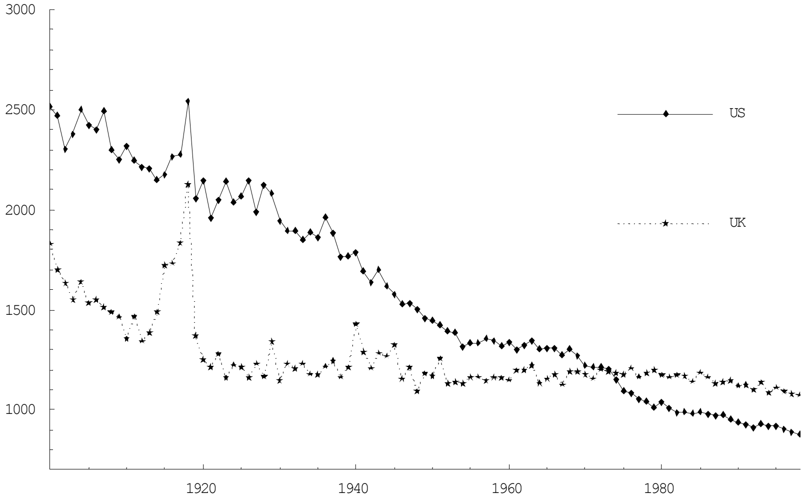
1900 – 1998 US and UK Population Death Rate per 100,000



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  - Ignore mortality correlation
- Improve existing mortality securitization models
  - Pricing the Swiss Re bond by multivariate exponential tilting with jump processes

Multivariate Exponential Tilting

Consider n variables $X_1, X_2, ..., X_n$ and k references Y_1, $Y_2, ..., Y_k$ on a probability space (Ω, P).

Definition 1. For each scenario ω in the probability space (Ω, P), the exponential tilting of $X_1, X_2, ..., X_n$ with respect to references Y_1, $Y_2, ..., Y_k$ is defined by the following p.d.f.:

$$\frac{f^*(x_1(\omega), x_2(\omega), ..., x_n(\omega))}{f(x_1(\omega), x_2(\omega), ..., x_n(\omega))} = c \left[\exp\left(\Sigma_{j=1}^k \lambda_j Y_j(\omega)\right)\right], \quad (3)$$

where $\lambda_1, \lambda_2, ..., \lambda_k$ are real-valued parameters that control the magnitude of risk-adjustment, and c is a normalizing coefficient.

Normalized Multivariate Exponential Tilting

Definition 2. Assume that there exist standard normal variables $Z_1, Z_2, ..., Z_k$ such that

$$Y_1 = F_{Y_1}^{-1}\left(\Phi(Z_1)\right), Y_2 = F_{Y_2}^{-1}\left(\Phi(Z_2)\right), ..., Y_k = F_{Y_k}^{-1}\left(\Phi(Z_k)\right). \tag{4}$$

Equation (4) can be obtained by percentile mapping.

$$\frac{f^*(x_1, x_2, ..., x_n)}{f(x_1, x_2, ..., x_n)} = c\mathrm{E}\left[\exp\left(\Sigma_{j=1}^{k}\lambda_j Z_j\right) | X_1 = x_1, X_2 = x_2, ..., X_n = x_n\right]. \tag{5}$$

Correlation Matrix

Theorem 1.

Assume that $\{X_1, X_2, ..., X_n; Y_1, Y_2, ..., Y_k\}$ follow a normal copula with correlation matrix:

$$\Sigma = \begin{pmatrix} \Sigma_{xx} & \Sigma_{xy} \\ \Sigma_{xy} & \Sigma_{yy} \end{pmatrix}. \tag{6}$$

Market Price of Risk

Theorem 1. (Cont'd)

The multivariate exponential tilting of $\{X_1, ..., X_n\}$ with respect to themselves ($Y_j = X_j, j = 1, ..., n$) is equivalent to the multivariate Wang transform (Wang, 2006) with

$$F^*_{X_i}(x_i) = \Phi[\Phi^{-1}(F_{X_i}(x_i)) + \beta_i],$$
$$\text{and } \beta_i = \sum_{j=1}^{n} \rho_{X_i,Y_j} \cdot \lambda_j, (\text{ for } i = 1, 2, ..., n). \quad (7)$$

The correlation matrix between $X_1, X_2, ..., X_n$ is unchanged after the normalized multivariate exponential tilting, $\Sigma^*_{xx} = \Sigma_{xx}$. Kijima (2006) reaches the same conclusion as equation (7) by using a multi-period equilibrium argument.

US and UK Population Index Dynamics

$$q^{us}_{t+h}|\mathcal{F}_t = q^{us}_t \exp\left[(\alpha^{us} - \frac{1}{2}\sigma^{us\ 2} - \Lambda^{us} k^{us} - \Lambda^{intl} k^{intl})h + \sigma^{us}\Delta W^{us}_t\right] \cdot \prod_{j>N^{us}_t}^{N^{us}_{t+h}} Y^{us}_j \prod_{i>N^{intl}_t}^{N^{intl}_{t+h}} Y^{intl}_i, \quad \text{and}$$

$$q^{uk}_{t+h}|\mathcal{F}_t = q^{uk}_t \exp\left[(\alpha^{uk} - \frac{1}{2}\sigma^{uk\ 2} - \Lambda^{uk} k^{uk} - \Lambda^{intl} k^{intl})h + \sigma^{uk}\Delta W^{uk}_t\right] \cdot \prod_{j>N^{uk}_t}^{N^{uk}_{t+h}} Y^{uk}_j \prod_{i>N^{intl}_t}^{N^{intl}_{t+h}} Y^{intl}_i, \quad \text{where } \underline{\underline{\mathrm{Cov}(W^{us}_t, W^{uk}_t) = \rho\sigma^{us}\sigma^{uk}}}.$$

Maximum Likelihood Parameter Estimates

- US and UK population mortality index 1900 – 1998.
- Maximum likelihood estimates:

α^{us}	-0.0100	α^{uk}	-0.0033		
σ^{us}	0.0308	σ^{uk}	0.0237	ρ	0.5299
Λ^{us}	10^{-6}	Λ^{uk}	0.8533	Λ^{intl}	0.0309
m^{us}	-0.0050	m^{uk}	-0.0114	m^{intl}	-0.0295
s^{us}	10^{-6}	s^{uk}	0.0600	s^{intl}	0.1412
k^{us}	-0.0050	k^{uk}	-0.0096	k^{intl}	-0.0193

Table: Maximum Likelihood Parameter Estimates Based on the US and UK Population Mortality Index 1900 –1998. The model without jumps is rejected at the significance level of 0.1%.

Correlation Matrix of US and UK Population Mortality Indices

- The correlation between W^{us} and W^{uk} is ρ.
- The jump sizes Y^{us}, Y^{uk} and Y^{intl} are independent of each other and of W^{us} and W^{uk}.
- Assume that $\{W^{us}, W^{uk}, Y^{us}, Y^{uk}, Y^{intl}\}$ use themselves as references.

According to **Theorem 1**,

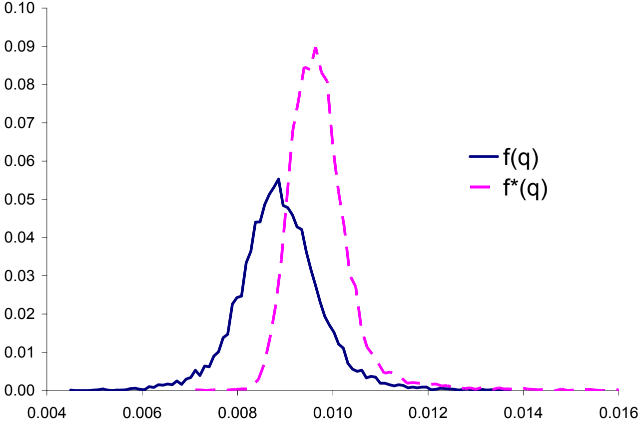
$$\Sigma^* = \Sigma = \begin{pmatrix} 1 & \rho & 0 & 0 & 0 \\ \rho & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0.5299 & 0 & 0 & 0 \\ 0.5299 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{8}$$

Market Price of Risk of Swiss Re Bond

$$
\begin{pmatrix} \beta_{W_{us}} \\ \beta_{W_{uk}} \\ \beta_{Y^{us}} \\ \beta_{Y^{uk}} \\ \beta_{Y^{intl}} \end{pmatrix} = \begin{pmatrix} 1 & \rho & 0 & 0 & 0 \\ \rho & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda_{W_{us}} \\ \lambda_{W_{uk}} \\ \lambda_{Y^{us}} \\ \lambda_{Y^{uk}} \\ \lambda_{Y^{intl}} \end{pmatrix}
$$

$$
= \begin{pmatrix} 1 & \rho & 0 & 0 & 0 \\ \rho & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda \\ \lambda \\ \lambda \\ \lambda \\ \lambda \end{pmatrix} = \begin{pmatrix} \lambda + \lambda\rho \\ \lambda + \lambda\rho \\ \lambda \\ \lambda \\ \lambda \end{pmatrix}. \tag{9}
$$

Transformed Distribution of $q = \max(q_{2004}, q_{2005}, q_{2006})$



Assume 80% on the US population mortality index and 20% on the UK Population mortality index.

Market Price of Risk of the Swiss Re Bond

	β_Y	β_W	Par Spread
Swiss Re Bond	**0.83**	**1.27**	1.35%

- Market price of risk for the property-linked catastrophe bond $\lambda = \mathbf{0.45}$ with $k = 6$ degrees of freedom.
- The Swiss Re overcompensates the investors for their taking its mortality risks.
  - Minton et al. (2004) conclude that securitization of financial institutions is a contracting innovation aimed at lowering financial distress costs.
  - MorganStanley (2003) concludes that "Swiss Re must be taking a view that the cost of capital that is relieved via this transaction exceeds the effective net cost of servicing the bond".
  - To develop mortality securitization markets

Conclusions

- Insurers develop mortality securitization market to manage catastrophic mortality risks.
- Improve existing mortality securitization models
  - Include jump process
  - Use multivariate exponential tilting
    - Correlation
    - Incomplete market pricing method
- Swiss Re bond seems a good deal for investors.

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