Multivariate Exponential Tilting and Pricing Implications for Mortality Securitization

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The Beauty of Risk Pooling

\[ \sigma_{\bar{X}} = \sigma_X / \sqrt{n} \]
The Curse of Risk Pooling

- Law of Large Numbers

\[
\text{Var}(\bar{T}) = \frac{1}{n} \text{Var}(T|\theta) \to 0. \tag{1}
\]

- Events that cause simultaneous effect

\[
\text{Var}(\bar{T}) = \mathbb{E}[\text{Var}(\bar{T}|\theta)] + \text{Var}\mathbb{E}(\bar{T}|\theta)
\]
\[
= \mathbb{E}\left[\frac{1}{n} \text{Var}(T|\theta)\right] + \text{Var}\mathbb{E}(T|\theta)
\]
\[
\to \text{Var}\mathbb{E}(T|\theta) \neq 0.
\]

- Pooling technique breaks down.
Shocks to a Life Insurer

1916–1920 US Female Population $q_x$

Human Mortality Database. University of California, Berkeley (USA), and Max Planck Institute for Demographic Research (Germany). Available at www.mortality.org or www.humanmortality.de (Data downloaded November 1–10, 2004).
Shocks to a Life Insurer (Cont’)

1916–1920 US Male Population $q_x$
## December 2004 Earthquake and Tsunami Death Toll and Percentage Excess Death Rates by Country

<table>
<thead>
<tr>
<th>Country</th>
<th>Confirmed deaths(^a)</th>
<th>Missing(^a)</th>
<th>% Excess Death Rate(^b)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Indonesia</td>
<td>127,420</td>
<td>116,368</td>
<td>16.58%</td>
</tr>
<tr>
<td>Sri Lanka</td>
<td>38,195</td>
<td>4,924</td>
<td>33.81%</td>
</tr>
<tr>
<td>India</td>
<td>10,779</td>
<td>5,614</td>
<td>0.18%</td>
</tr>
<tr>
<td>Thailand</td>
<td>5,395</td>
<td>2,991</td>
<td>1.90%</td>
</tr>
<tr>
<td>Somalia</td>
<td>298</td>
<td>-</td>
<td>0.21%</td>
</tr>
<tr>
<td>Myanmar</td>
<td>90</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Maldives</td>
<td>82</td>
<td>-</td>
<td>3.25%</td>
</tr>
<tr>
<td>Malaysia</td>
<td>68</td>
<td>-</td>
<td>0.06%</td>
</tr>
<tr>
<td>Tanzania</td>
<td>10</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Bangladesh</td>
<td>2</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Kenya</td>
<td>1</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>182,340</strong></td>
<td><strong>129,897</strong></td>
<td><strong>-</strong></td>
</tr>
</tbody>
</table>

\(^a\)Source: Associated Press on March 4, 2005; \(^b\)Based on the authors’ calculation.
Shocks to an Annuity Insurer/Pension Plan

- Dramatic mortality improvement
- Increased demand for annuities (Mitchell et al., 2001)
- Private DB pension plans
- Existing social security system in the US
Life Insurance-linked Cat Securitization

- Multiple risks
  - Interest rate risk
  - Policyholder lapse risk
  - Regulatory risk
  - Mortality risk
  - Insurer policy dividend decisions

- Pure mortality or longevity risk
  - Increase transparency (Cowley and Cummins, 2005)
  - Diversification benefit (Lin and Cox, 2005)
Swiss Re Deal

- Issued December 2003, matures January 1, 2007, a three year deal.
- No coupons at risk
- Priced to sell at par with a coupon of LIBOR + 1.35%.
- Principal at risk. $q$ = weighted average population mortality in US (70%), UK (15%), France (7.5%), Italy (5%), and Switzerland (2.5%). $q_0 = 2002$ level and $q = \max(q_{2004}, q_{2005}, q_{2006})$

\[
\text{Maturity value} = \begin{cases} 
400,000,000 & \text{if } q \leq 1.3q_0 \\
400,000,000 \frac{1.5q_0 - q}{0.2q_0} & \text{if } 1.3q_0 < q \leq 1.5q_0 \\
0 & \text{if } q > 1.5q_0 
\end{cases}
\]
Mortality Securitization Modeling Literature

- Existing mortality securitization pricing theory
  - Ignore mortality jumps
    - Itô–type stochastic process (Dahl, 2003; Milevsky and Promislow, 2001; Cairns et al., 2004)
    - Econometric methods (Renshaw et al., 1996; Sithole et al., 2000; Lee and Carter, 1992; Lee, 2000)
1900 – 1998 US Population Death Rate per 100,000
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- Improve existing mortality securitization models
  - Pricing the Swiss Re bond by multivariate exponential tilting with jump processes
Consider $n$ variables $X_1, X_2, \ldots, X_n$ and $k$ references $Y_1, Y_2, \ldots, Y_k$ on a probability space $(\Omega, P)$.

**Definition 1.** For each scenario $\omega$ in the probability space $(\Omega, P)$, the exponential tilting of $X_1, X_2, \ldots, X_n$ with respect to references $Y_1, Y_2, \ldots, Y_k$ is defined by the following p.d.f.:

$$
\frac{f^*(x_1(\omega), x_2(\omega), \ldots, x_n(\omega))}{f(x_1(\omega), x_2(\omega), \ldots, x_n(\omega))} = c \left[ \exp \left( \sum_{j=1}^{k} \lambda_j Y_j(\omega) \right) \right],
$$

(3)

where $\lambda_1, \lambda_2, \ldots, \lambda_k$ are real-valued parameters that control the magnitude of risk-adjustment, and $c$ is a normalizing coefficient.
Definition 2. Assume that there exist standard normal variables $Z_1, Z_2, \ldots, Z_k$ such that

$$Y_1 = F_{Y_1}^{-1} (\Phi(Z_1)), \ Y_2 = F_{Y_2}^{-1} (\Phi(Z_2)), \ldots, Y_k = F_{Y_k}^{-1} (\Phi(Z_k)).$$

(4)

Equation (4) can be obtained by percentile mapping.

$$\frac{f^*(x_1, x_2, \ldots, x_n)}{f(x_1, x_2, \ldots, x_n)} = c \mathbb{E} \left[ \exp \left( \sum_{j=1}^{k} \lambda_j Z_j \right) | X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n \right].$$

(5)
Theorem 1.

Assume that \( \{X_1, X_2, \ldots, X_n; Y_1, Y_2, \ldots, Y_k\} \) follow a normal copula with correlation matrix:

\[
\Sigma = \begin{pmatrix}
\Sigma_{xx} & \Sigma_{xy} \\
\Sigma_{xy} & \Sigma_{yy}
\end{pmatrix}.
\]
Theorem 1. (Cont’d)

The multivariate exponential tilting of \( \{X_1, \ldots, X_n\} \) with respect to themselves \( (Y_j = X_j, j = 1, \ldots, n) \) is equivalent to the multivariate Wang transform (Wang, 2006) with

\[
F^*_{X_i}(x_i) = \Phi[\Phi^{-1}(F_{X_i}(x_i)) + \beta_i],
\]

and \( \beta_i = \sum_{j=1}^{n} \rho_{X_i,Y_j} \cdot \lambda_j, \) (for \( i = 1, 2, \ldots, n \)). \hspace{1cm} (7)

The correlation matrix between \( X_1, X_2, \ldots, X_n \) is unchanged after the normalized multivariate exponential tilting, \( \Sigma^*_xx = \Sigma_{xx} \). Kijima (2006) reaches the same conclusion as equation (7) by using a multi-period equilibrium argument.
US and UK Population Index Dynamics

\[ q_{t+h}^{us} | \mathcal{F}_t = q_t^{us} \exp \left[ \left( \alpha^{us} - \frac{1}{2} \sigma^{us} 2 - \Lambda^{us} k^{us} - \Lambda^{intl} k^{intl} \right) h + \sigma^{us} \Delta W_t^{us} \right] \]

\[ \cdot \prod_{j > N_t^{us}} Y_j^{us} \prod_{i > N_t^{intl}} Y_i^{intl}, \quad \text{and} \]

\[ q_{t+h}^{uk} | \mathcal{F}_t = q_t^{uk} \exp \left[ \left( \alpha^{uk} - \frac{1}{2} \sigma^{uk} 2 - \Lambda^{uk} k^{uk} - \Lambda^{intl} k^{intl} \right) h + \sigma^{uk} \Delta W_t^{uk} \right] \]

\[ \cdot \prod_{j > N_t^{uk}} Y_j^{uk} \prod_{i > N_t^{intl}} Y_i^{intl}, \quad \text{where} \quad \text{Cov} (W_t^{us}, W_t^{uk}) = \rho \sigma^{us} \sigma^{uk}. \]
Maximum Likelihood Parameter Estimates


- Maximum likelihood estimates:

<table>
<thead>
<tr>
<th></th>
<th>α&lt;sup&gt;us&lt;/sup&gt;</th>
<th>α&lt;sup&gt;uk&lt;/sup&gt;</th>
<th>σ&lt;sup&gt;us&lt;/sup&gt;</th>
<th>σ&lt;sup&gt;uk&lt;/sup&gt;</th>
<th>ρ</th>
<th>Λ&lt;sup&gt;us&lt;/sup&gt;</th>
<th>Λ&lt;sup&gt;uk&lt;/sup&gt;</th>
<th>Λ&lt;sup&gt;intl&lt;/sup&gt;</th>
<th>m&lt;sup&gt;us&lt;/sup&gt;</th>
<th>m&lt;sup&gt;uk&lt;/sup&gt;</th>
<th>m&lt;sup&gt;intl&lt;/sup&gt;</th>
<th>s&lt;sup&gt;us&lt;/sup&gt;</th>
<th>s&lt;sup&gt;uk&lt;/sup&gt;</th>
<th>s&lt;sup&gt;intl&lt;/sup&gt;</th>
<th>k&lt;sup&gt;us&lt;/sup&gt;</th>
<th>k&lt;sup&gt;uk&lt;/sup&gt;</th>
<th>k&lt;sup&gt;intl&lt;/sup&gt;</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>-0.0100</td>
<td>-0.0033</td>
<td>0.0308</td>
<td>0.0237</td>
<td>0.5299</td>
<td>10⁻⁶</td>
<td>0.8533</td>
<td>0.0309</td>
<td>-0.0050</td>
<td>-0.0114</td>
<td>-0.0295</td>
<td>10⁻⁶</td>
<td>0.0600</td>
<td>0.1412</td>
<td>-0.0050</td>
<td>-0.0096</td>
<td>-0.0193</td>
</tr>
</tbody>
</table>

**Table:** Maximum Likelihood Parameter Estimates Based on the US and UK Population Mortality Index 1900 –1998. The model without jumps is rejected at the significance level of 0.1%.
Correlation Matrix of US and UK Population Mortality Indices

- The correlation between $W^{us}$ and $W^{uk}$ is $\rho$.
- The jump sizes $Y^{us}$, $Y^{uk}$ and $Y^{intl}$ are independent of each other and of $W^{us}$ and $W^{uk}$.
- Assume that $\{W^{us}, W^{uk}, Y^{us}, Y^{uk}, Y^{intl}\}$ use themselves as references.

According to Theorem 1,

$$\Sigma^* = \Sigma = \begin{pmatrix}
1 & \rho & 0 & 0 & 0 \\
\rho & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix} = \begin{pmatrix}
1 & 0.5299 & 0 & 0 & 0 \\
0.5299 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}.$$ (8)
Market Price of Risk of Swiss Re Bond

\[
\begin{pmatrix}
\beta_{W_{us}} \\
\beta_{W_{uk}} \\
\beta_{Y_{us}} \\
\beta_{Y_{uk}} \\
\beta_{Y_{intl}}
\end{pmatrix} = 
\begin{pmatrix}
1 & \rho & 0 & 0 & 0 \\
\rho & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix} 
\begin{pmatrix}
\lambda_{W_{us}} \\
\lambda_{W_{uk}} \\
\lambda_{Y_{us}} \\
\lambda_{Y_{uk}} \\
\lambda_{Y_{intl}}
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & \rho & 0 & 0 & 0 \\
\rho & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix} 
\begin{pmatrix}
\lambda \\
\lambda \\
\lambda \\
\lambda \\
\lambda
\end{pmatrix} = 
\begin{pmatrix}
\lambda + \lambda \rho \\
\lambda + \lambda \rho \\
\lambda \\
\lambda \\
\lambda
\end{pmatrix}.
\]
Assume 80% on the US population mortality index and 20% on the UK Population mortality index.
Market Price of Risk of the Swiss Re Bond

<table>
<thead>
<tr>
<th></th>
<th>$\beta_Y$</th>
<th>$\beta_W$</th>
<th>Par Spread</th>
</tr>
</thead>
<tbody>
<tr>
<td>Swiss Re Bond</td>
<td>0.83</td>
<td>1.27</td>
<td>1.35%</td>
</tr>
</tbody>
</table>

- Market price of risk for the property-linked catastrophe bond $\lambda = 0.45$ with $k = 6$ degrees of freedom.
- The Swiss Re overcompensates the investors for their taking its mortality risks.
  - Minton et al. (2004) conclude that securitization of financial institutions is a contracting innovation aimed at lowering financial distress costs.
  - MorganStanley (2003) concludes that “Swiss Re must be taking a view that the cost of capital that is relieved via this transaction exceeds the effective net cost of servicing the bond”.
- To develop mortality securitization markets
Conclusions

- Insurers develop mortality securitization market to manage catastrophic mortality risks.
- Improve existing mortality securitization models
  - Include jump process
  - Use multivariate exponential tilting
    - Correlation
    - Incomplete market pricing method
- Swiss Re bond seems a good deal for investors.


