Killing the Law of Large Numbers:

Financial Valuation of Mortality Risk via the Instantaneous Sharpe Ratio

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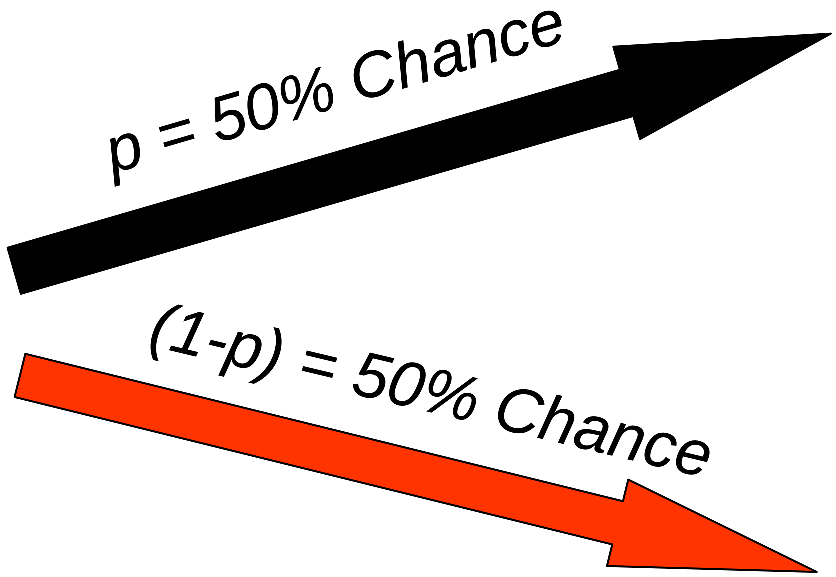
Longevity Risk Symposium
24 April 2006, Chicago

Outline & Agenda

- Basic review of how the LLN breaks down under aggregate mortality risk and why this impacts risk management.
- Discuss the existing academic literature.
- Introduce our Sharpe Ratio based approach for pricing longevity-linked instruments in discrete time.
- Brief overview of how to model this in continuous-time…(time permitting)

Longevity Insurance Payoff:



Pay:
$(1+L)$

p = 50% Chance

Pays $2
Alive

(1-p) = 50% Chance

Pays $0
Dead

Back to Classical Basics:

$$w_i = \begin{cases} 2 & \Pr = (p) \\ 0 & \Pr = (1-p) \end{cases}$$

$$E[w_i] = 2p$$

$$\mathrm{var}[w_i] = p(2-2p)^2 + (1-p)(0-2p)^2$$

$$= 4p(1-p)$$

Back to Classical Basics:

$$w_i = \begin{cases} 2 & \Pr = 0.5 \\ 0 & \Pr = 0.5 \end{cases}$$

$$E[w_i] = 1$$

$$\mathrm{var}[w_i] = 1$$

$$SD[w_i] = 1$$

Adding-up the longevity bets:

What is total payout when selling N policies?

$$W = \sum_{i=1}^{N} w_i$$

Term is larger when risks are dependent

$$E[W] = E\left[\sum_{i=1}^{N} w_i\right] = N2p$$

$$\mathrm{var}[W] = \mathrm{var}\left[\sum_{i=1}^{N} w_i\right] = N4p(1-p)$$

Refresher: Law of Large Numbers: Under i.i.d. payouts…

$$\lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} w_i \to E[w_i]$$

Within the context of insurance this implies that eventually there is no uncertainty (i.e. risk) in what you will be paying out per policy and the payout risk is not compensated in equilibrium. In the language of portfolio theory, it is diversifiable.

The idiosyncratic Risk which is the total standard deviation per policy, goes to zero…

$$\frac{1}{N}SD[W]=\frac{2\sqrt{p(1-p)}}{\sqrt{N}}\rightarrow 0$$

You expect each policy to generate a payout of $1 with
zero uncertainty (risk), as long as you sell enough policies.

…I can also use the Central Limit Theorem (CLT) to compute probabilities

$$\Pr[W \le C] = \Pr\left[\frac{W - E[W]}{SD[W]} \le \frac{C - E[W]}{SD[W]}\right]$$

$$\approx \Pr\left[Z \le \frac{C - 2Np}{\sqrt{4Np(1-p)}}\right]$$

Or I can go thru the brute-force way using the Binomial distribution.
More on this later…

Numerical Example:
Sell 10,000 Longevity Insurance Policies

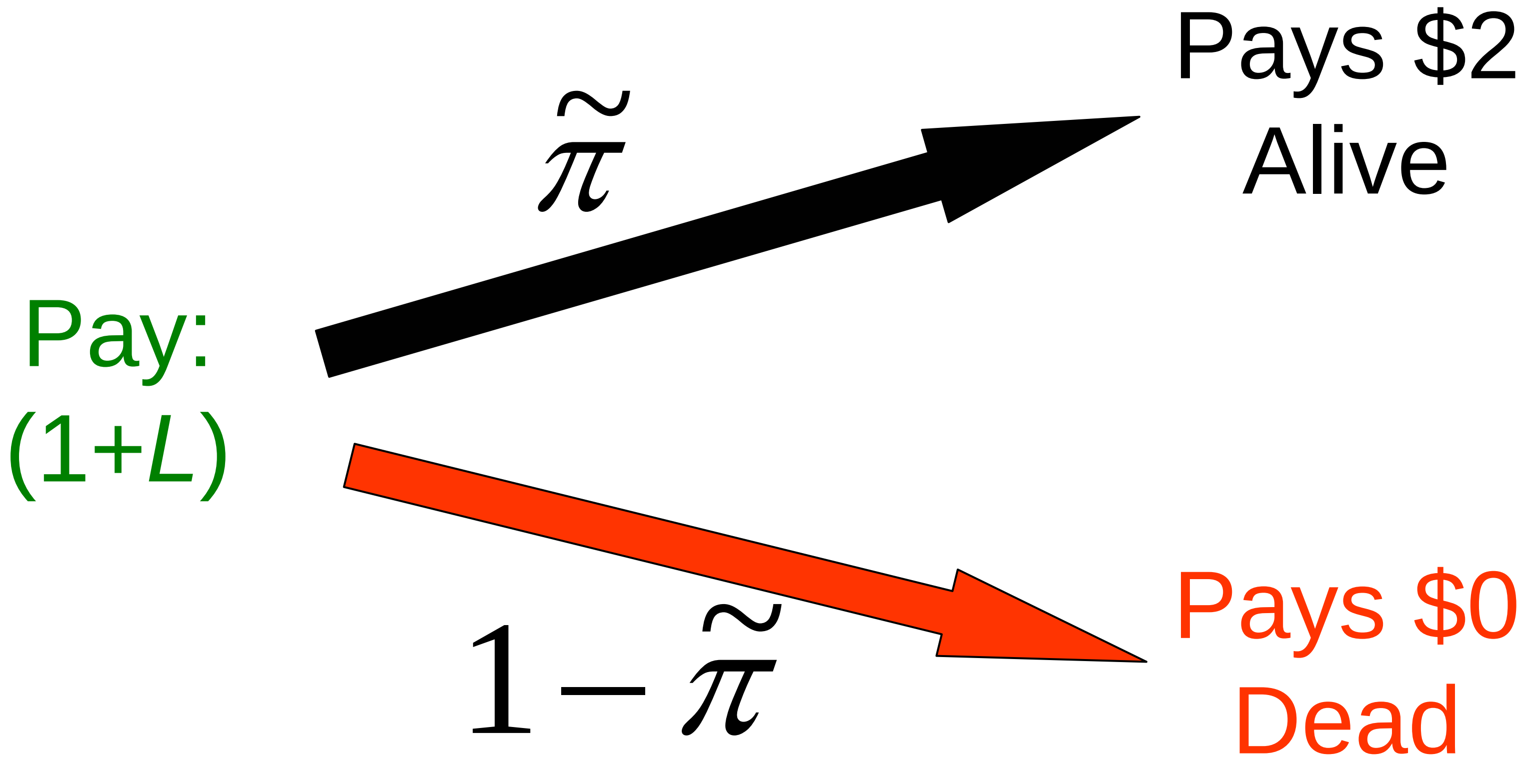
Total Payout Outside the Range of:	Event Probability:
(\$5,000 -- \$15,000)	0.000%
(\$7,500 -- \$12,500)	0.000%
(\$9,000 -- \$11,000)	0.000%
(\$9,500 -- \$10,500)	0.000%
(\$9,700 -- \$10,300)	0.271%

Note: Standard deviation of payout
is only \$100, since p = 0.5.

What happens under stochastic hazard parameters…

- The first (classical) example assumed we know the parameter value p with perfect certainty.
- What if we are unsure of p? Are we entitled to simply compute the expectation *E[p]* and use that estimate for pricing & risk management formulas?
- **No!**
- In fact, the total payout distribution becomes a mixture of Binomials, the Law of Large Numbers (LLN) -- invoked for pricing insurance -- breaks down and some risk can never be eliminated.

Longevity Insurance Payoff: Stochastic Hazard Parameters



Pay:
(1+*L*)

$\tilde{\pi}$

Pays $2
Alive

$1-\tilde{\pi}$

Pays $0
Dead

Modern Basics (II): Stochastic Hazard Parameters

$$w^*_i = \begin{cases} 2 & \Pr = \tilde{\pi} \\ 0 & \Pr = 1 - \tilde{\pi} \end{cases}$$

$$\tilde{\pi} = \begin{cases} \pi_1 & \Pr = (p_1) \\ \pi_2 & \Pr = (p_2) \end{cases}$$

$$E[\tilde{\pi}] = \pi_1 p_1 + \pi_2 p_2$$

Modern Basics (II): Example
Stochastic Hazard Parameters

$$w^{*}_{i} = \begin{cases} 2 & \Pr = \tilde{\pi} \\ 0 & \Pr = 1 - \tilde{\pi} \end{cases}$$

$$\tilde{\pi} = \begin{cases} \pi_1 = 0.6 & \Pr = 0.5 \\ 1 - \pi_1 = 0.4 & \Pr = 0.5 \end{cases}$$

$$E[\tilde{\pi}] = 0.5$$

-> Which was the assumed survival probability in the traditional case

Adding-up the Longevity bets…
under Stochastic Hazard Parameters

$$W^* = \sum_{i=1}^{N} w^*{}_i$$

$$E[W^*] = E\left[\sum_{i=1}^{N} w^*{}_i\right] = N2E[\tilde{\pi}]$$

Adding-up the Longevity bets… under Stochastic Hazard Parameters

$$W^{*}=\sum_{i=1}^{N}w^{*}{}_{i}$$

$$E[W^{*}]=E\left[\sum_{i=1}^{N}w^{*}{}_{i}\right]=N2E[\tilde{\pi}]$$

$$\mathrm{var}[W^{*}]=E[\mathrm{var}[W^{*}\,|\,\pi]]+\mathrm{var}[E[W^{*}\,|\,\pi]]$$

Decomposing the Payout Variance

$$E[\mathrm{var}[W^* \mid \pi]] = 4N\pi_1(1-\pi_1)$$

$$\mathrm{var}[E[W^* \mid \pi]] = 4N^2 p_1(1-p_1)(2\pi_1-1)^2$$

The total (payout) variance is the sum of these two components.
Note that the second portion would be zero under the classical approach.

Decomposing the Payout Variance: A Numerical Example

$$E[\text{var}[W^* \mid \pi]] = 4N(0.6)(0.4)$$

$$= 0.96N$$

$$\text{var}[E[W^* \mid \pi]] = 4N^2(0.5)(0.5)(0.2)^2$$

$$= 0.04N^2$$

$$\text{var}[W^*] = 0.96N + 0.04N^2$$

Notice that when N = 1, the variance equals $1 which
is the same as classical case. A single policy is not riskier!
But, when N gets large we have a problem…

In this case the Risk per policy does not go to zero which is our main problem and motivation…

$$\lim_{N\to\infty}\frac{1}{N}SD[W^*]\to 2(2\pi_1-1)\sqrt{p_1(1-p_1)}\neq 0$$

When $\pi_1 = 0.5$, which implies no parameter uncertainty, the expression does go to zero. The same idea applies when p_1 or p_2 equal zero. The classical model is a special case.

How Fast Does Risk Decline?
Total Standard Deviation per Policy

Number of Policies Sold: **N**	**Deterministic Hazard Parameter** **$p = 0.5$**	**Stochastic Hazard Parameter** **$E[\pi] = 0.5$**
1	$1.000	$1.000
2	$0.707	$0.721
5	$0.447	$0.482
100	$0.100	$0.223
1000	$0.032	$0.202
10,000	$0.010	$0.200
Infinity	$0.000	$0.200

$\pi = 0.6$ or 0.4 with even odds

The Mixture of Binomial Distributions: Another way to get the same results

$$\Pr[W \leq 2k] = \sum_{i=0}^{k} B(i \mid N, p)$$

$$\Pr[W^{*} \leq 2k] =$$

$$\sum_{i=0}^{k} p_1 B(i \mid N, \pi_1) + (1 - p_1) B(i \mid N, 1 - \pi_1)$$

Where:

$$B(i \mid N, p) := \binom{N}{i} p^{i} (1 - p)^{N-i}$$

You sold 100 Longevity Insurance policies: Comparing the two cases:

Total Payout Larger Than:	$\Pr[W > C]$	$\Pr[W^* > C]$
	Diversifiable	**Non-Diversifiable**
$102	0.382	0.483
$110	0.136	0.411
$120	0.018	0.231
$130	0.001	0.065

Note: Survivor gets $2

$E[\pi] = (0.6)(0.5)+(0.4)(0.5) = p = 0.5$

A Quick Break:

Review Previous & Current Literature…

- Lee & Carter (1992), Olivieri (2001)
- Milevsky & Promislow (2001), Dahl (2004), Biffis (2005), Schrager (2006), Ballotta & Haberman (2006), Biffis & Millossovich (2006).
- Denuit & Dhaene (2006).
- Cairns, Blake & Dowd (2005), Cox & Lin (2004), Webb & Friedberg (2006).

Refresher on the Sharpe Ratio:

Common financial language used in thinking about compensation for market risk.

$$\alpha := \frac{E[X] - R}{SD[X]}$$

$$\left.\begin{aligned} E[X] &= 0.11 \\ R &= 0.06 \\ SD[X] &= 0.20 \end{aligned}\right\} \quad \alpha = 0.25$$

Roughly in-line with historical returns for equity markets.

Our contribution:

(Think about) pricing via the Sharpe Ratio

- How much would you charge for the longevity insurance policy if you wanted to be compensated for this (non diversifiable) risk in proportion to the Sharpe Ratio?
- As a crude example, if your standard deviation per policy is converging to \$0.20 and you demand a Sharpe Ratio of $\alpha = 0.25$, then you would charge a loading of **L** = 5% above the risk-free rate.
- Yes. This is just another premium principle.

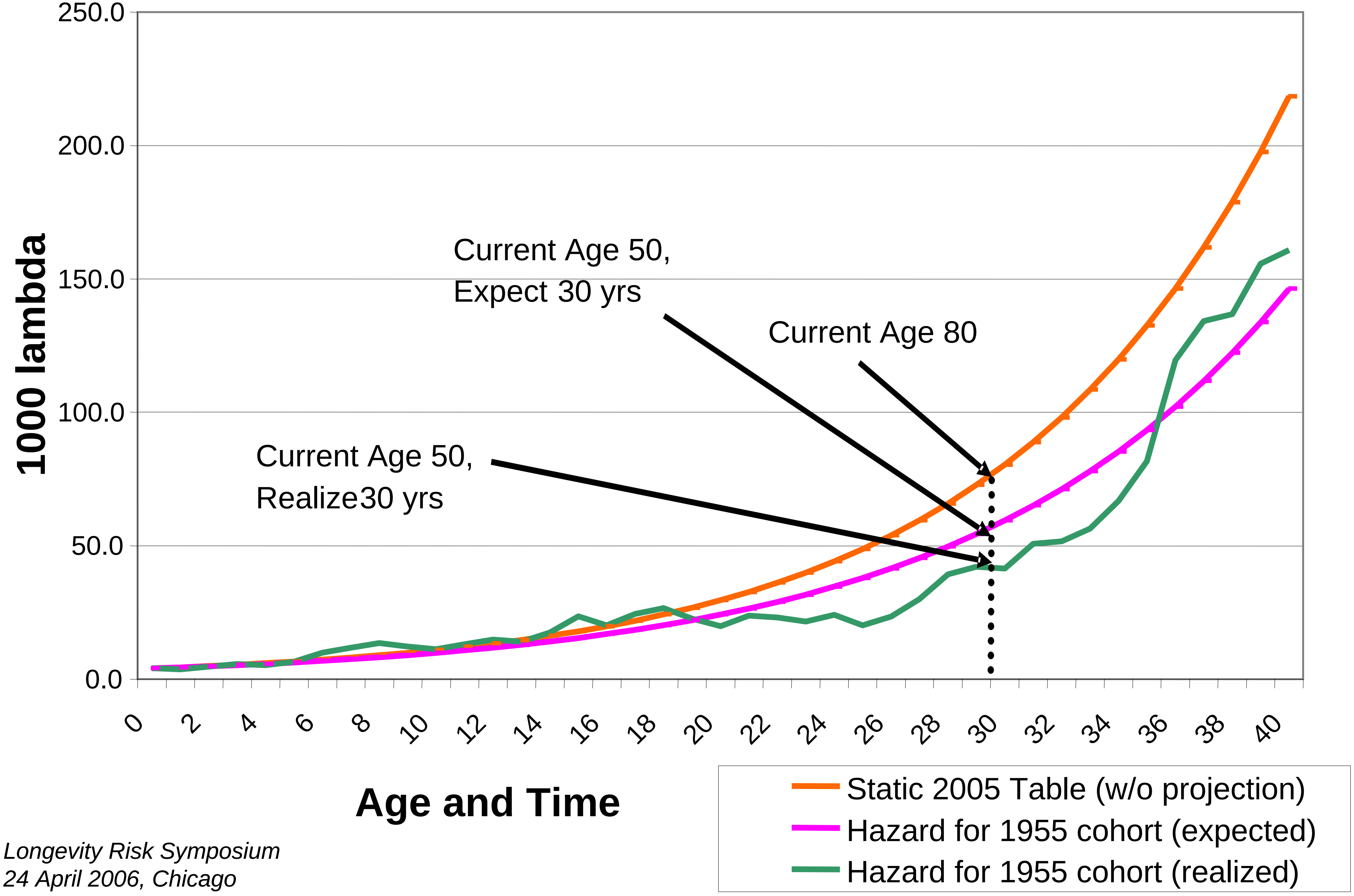
Ok. How does this model work in continuous time?

$$d\lambda_t = a(\lambda_t, t)dt + b(t)(\lambda_t - \underline{\lambda})dW_t^{\lambda}$$

$$dr_t = \mu(r_t, t)dt + \sigma(r_t, t)dW_t$$

The variable λ denotes the Instantaneous Force of Mortality (IFM)
which is akin to the stochastic hazard parameter π in the discrete case.
It is being driven by a Brownian motion (i.e. a random walk).

Diffusion Hazard Rates


Current Age 50,
Expect 30 yrs

Current Age 80

Current Age 50,
Realize30 yrs

Financial Perspective on Mortality:

Do Not Confuse The Two

$$E^{Q}[\lambda_t]$$

Financial Economic

$$E^{P}[\lambda_t]$$

Bio-statistical

Remember that if a mortality risk premium exists…

$$E^{Q}[\lambda_t] \neq E^{P}[\lambda_t]$$

One Possible Hazard Rate Process: Milevsky & Promislow (2001)

$$\lambda_t = \lambda_0 \exp\{gt + \xi Y_t\}$$

$$dY_t = -\kappa Y_t + dB_t$$

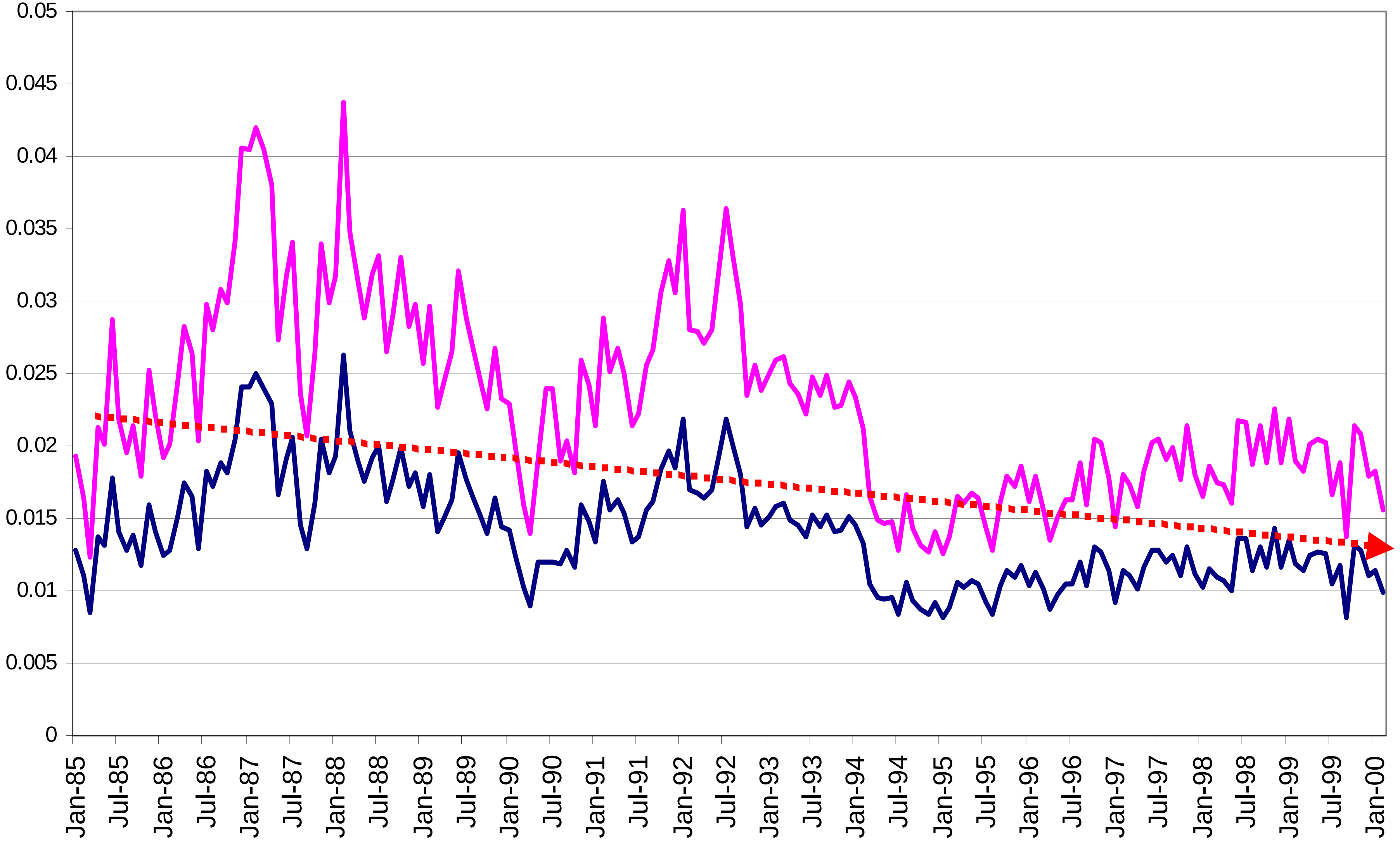
Mean Reverting Brownian-driven Gompertz (MRBG) model.
Related to the Black-Derman-Toy Model for interest rates.

Calibrating Mortality Diffusions Using Annuity Payouts

Pc\Age	**55**	**60**	**65**	**70**	**75**	**80**
0	631	686	765	877	1039	1259
5	628	681	755	855	989	1146
10	620	666	726	799	879	940
15	607	644	687	729	764	774
20	591	618	643	662	673	665
25	573	589	601	608	610	N.A.

Source: CANNEX Financial 2002 (unisex)
Monthly income per $100,000 premium

The Evolution of Implied Hazard Rates
Age 65: 1985 - 2000



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Back to our story:

Hedging Portfolio for a Pure Endowment Policy

$$\Pi_t = -P(r_t, \lambda_t, t) + \pi_t F(r_t, t)$$

Sell the pure endowment (longevity insurance) for **P** and then
hedge the risk using a portfolio of risk-free bonds denoted by **F**.
In this set-up the variable π denotes the hedge ratio.

Local Standard Deviation (Instantaneous Sharpe Ratio) of the Hedging Portfolio

$$\lim_{h \to 0} \sqrt{\frac{1}{h} \operatorname{var}(\Pi_{t+h} \mid \mathrm{F}_t)}$$

$$= \sqrt{b^2(t)(\lambda - \underline{\lambda})^2 P_\lambda{}^2(r, \lambda, t) + \lambda P^2(r, \lambda, t)}$$

We value (price) the pure endowment by setting the drift of the hedging portfolio equal to the short rate ***r****, times portfolio value plus* α *times the local standard deviation.*

Value (price) of the claim solves:

$$P_t + \mu^Q P_r + \frac{1}{2}\sigma^2 P_{rr} + aP_\lambda + \frac{1}{2}b^2(\lambda - \underline{\lambda})^2 P_{\lambda\lambda} - (r+\lambda)P$$

$$= -\alpha\sqrt{b^2(\lambda - \underline{\lambda})^2 P_\lambda{}^2 + \lambda P^2}$$

$$P(r,\lambda,T) = 1$$

Our paper proves that P satisfies a number of appealing properties:

(1) It is sub-additive in the number of contracts sold.

(2) The pricing survival probability is greater than the physical probability

Practical Conclusion

- Most researchers (and practitioners) now agree that there is a mortality risk premium.
- At a minimum this means we – as instructors -- should change the way we teach (life, annuity) insurance pricing to students.
- We also believe the concept of Sharpe Ratios can be helpful in understanding the risk & reward tradeoff, as well as linking these ideas to financial economics.
- Slides & updated paper at www.ifid.ca