A TWO-FACTOR MODEL FOR STOCHASTIC MORTALITY WITH PARAMETER UNCERTAINTY

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Plan

● Introduction + data

● A two-factor model for stochastic mortality

● Applications
  – Longevity bonds
  – Survivor caps and caplets

● Conclusions
The facts about mortality:

- Life expectancy is increasing.
- Future development of life expectancy is uncertain.

"Longevity risk"

Longevity Risk = the risk that future mortality rates are lower than anticipated

Focus here: Mortality rates above age 60
England and Wales log mortality rates 1950-2002

Age 60

Age 80
Stochastic Models

Limited historical data ⇒

- No single model is ‘the right one’
- Parameter risk
- Model risk
Case study: England and Wales males, age 60-95

$q_y = \text{mortality rate at age } y \text{ in } 2002$

Data suggests $\log \frac{q_y}{(1 - q_y)}$ is linear
A TWO-FACTOR ‘SHORT-RATE’ MODEL

Cohort: Age $x$ at time $t = 0$

Mortality rates for the year $t$ to $t + 1$:

$$q(t, x) = \frac{e^{A_1(t) + A_2(t)(x+t)}}{1 + e^{A_1(t) + A_2(t)(x+t)}}$$

$(x + t) = \text{age at time } t$

We model $A(t) = (A_1(t), A_2(t))^\prime$ as a random-walk with drift
\[ A(t) = (A_1(t), A_2(t))' \]

Model: Random walk with drift

\[ A(t + 1) - A(t) = \mu + CZ(t + 1) \]

- \( V = CC' \) = variance-covariance matrix
- Estimate \( \mu \) and \( V \)
- Quantify parameter uncertainty in \( \mu \) and \( V \)
Application: longevity bonds

- Cohort: Age $x$ at time $t = 0$

- $S(t, x) = \text{survivor index at } t$
  
  proportion surviving from time 0 to time $t$

\[
S(t, x) = (1 - q(0, x)) \times \ldots \times (1 - q(t - 1, x))
\]
90% Confidence Interval (CI) for Cohort Survivorship

Data from 1982–2002

Proportion surviving, $S(x)$

- $E[S(x)]$ with param. uncertainty
- CI without param. uncertainty
- CI with param. uncertainty

Age
Stochastic Mortality: General Conclusions

- Less than 10 years:
  - Systematic risk not significant

- Over 10 years
  - Systematic risk becomes more and more significant over time

- Over 20 years
  - Model and parameter risk begin to dominate
Risk-neutral pricing

\[
\begin{pmatrix}
A_1(t + 1) \\
A_2(t + 1)
\end{pmatrix}
= 
\begin{pmatrix}
A_1(t) \\
A_2(t)
\end{pmatrix} + 
\begin{pmatrix}
\mu_1 \\
\mu_2
\end{pmatrix}
+ 
\begin{pmatrix}
c_{11} & c_{12} \\
c_{21} & c_{22}
\end{pmatrix}
\begin{pmatrix}
\tilde{Z}_1(t + 1) + \lambda_1 \\
\tilde{Z}_2(t + 1) + \lambda_2
\end{pmatrix}
\]

where \(\tilde{Z}_1(t + 1)\) and \(\tilde{Z}_2(t + 1)\) are i.i.d. \(\sim N(0, 1)\)

under a risk-neutral pricing measure \(Q(\lambda)\)

\(\lambda_1\) and \(\lambda_2\) are market prices of risk
Comments

- The market is highly incomplete
- The switch from $P$ to $Q$ is a modelling assumption
- (Simple) Key assumption:
  
  market prices of risk $\lambda_1$ and $\lambda_2$ are constant.
- As a market develops this assumption becomes a testable hypothesis
One data point: the EIB-BNP longevity bond

- Offer price (ultimately unsuccessful) ⇒
  risk premium of 20 basis points if held to maturity

- What values of $\lambda_1$, $\lambda_2$ are consistent with the 20b.p.’s risk premium?

- One price, two parameters ⇒ many solutions
Answer: 20 b.p. spread equates to

\[ \lambda_1 = 0.375, \quad \lambda_2 = 0 \]

\[ \downarrow \quad \downarrow \]

\[ \lambda_1 = 0, \quad \lambda_2 = 0.315 \]

Do these values represent a *good deal*?

Why do we need to know \( \lambda_1, \lambda_2 \)?

\[ \Rightarrow \text{ info. on how to price new issues in the future.} \]
Longevity Bond Risk Premiums: \( \lambda = (0.375, 0) \)

Dependency on term and initial age:

<table>
<thead>
<tr>
<th></th>
<th>Initial age of cohort, ( x )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>60</td>
</tr>
<tr>
<td>Bond 20</td>
<td>8.9</td>
</tr>
<tr>
<td>Maturity 25</td>
<td>12.7</td>
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<tr>
<td>( T ) 30</td>
<td>16.9</td>
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</tbody>
</table>
At-the-Money Call Options on $S(t)$

Payoff: $\max\{S(T) - K, 0\}$ where $K = E_{Q(\lambda)}[S(T)]$

25-year survivor cap: parameter uncertainty adds 33%
Conclusions

- Stochastic models important for
  - risk measurement → assessment of risk premium
  - pricing contracts with option characteristics
- One model out of many possibilities
- **Significant** longevity risk in the medium/long term
- Model and parameter risk is important
Selected References


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